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Mathematical Inequalities

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MATHEMATICAL INEQUALITIES

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ABSTRACT

MATHEMATICAL INEQUALITIES

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In this thesis, we discuss mathematical inequalities, which arise in various branches of Mathematics and other related fields. The subject is a vast one, but our focus is on inequalities related to complex analysis, geometry, and matrix theory.

We investigate recently proven trigonometric and hyperbolic inequalities. This includes Katsuura's string of seven inequalities for the sine and tangent functions and Price's Inequality (with new proofs derived by Katsuura and Obaid). We also discuss complex hyperbolic inequalities and inequalities from infinite products.

We then establish geometric inequalities, including those relating parts of the triangle as well as conic sections and their tangent lines. We also develop proofs of the Arithmetic-Geometric Mean and Erdös-Mordell inequalities.

Finally, we explore inequalities for univalent functions, including the famous Bieberbach Conjecture, Area Theorem, and Koebe's One-Quarter Theorem. We finish with Hadamard's Inequality for determinants

DEDICATION

I would like to dedicate my thesis to my wonderful advisor, Dr. Samih Obaid, to whom I am forever grateful. Without him and his constant support, this thesis would not have been possible and the process not nearly as enjoyable. Dr. Obaid is one of the kindest people I have ever met, and his ingenuity, hard work, and positive outlook are absolutely inspiring. I spent every one of our meetings engaged in Mathematics and energized about my topic. And no matter how overwhelmed or stressed I was, he was always able to make me laugh. Thank you, Dr. Obaid, for showing me what it means to be a great mathematician, teacher, and friend.

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To my family – Dad, you've shown me what it means to work hard and never give up; Mom, you are the most supportive someone could be; and RJ, you've taught me to not take life so seriously. I am truly blessed to have you for my family.

And finally, to my boyfriend, Steve. Thank you for not only believing in me, but for putting up with me throughout this whole process. I couldn't have done it without your patience, understanding, and sense of humor.

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INTRODUCTION

Mathematical inequalities are essential to the study of Mathematics as well as many related fields, and their uses are extensive. The database of the American Mathematical Society includes more than 23,000 references of inequalities and their applications. While the concept is a simple one, some of the most famous and significant results in Mathematics are inequalities.

There are many classical inequalities that are not only well known but also quite relevant today. For instance, Schur's Inequality states that, for any positive real numbers a, b, and c, and real number r, it is always the case that

$$a^r(a-b)(a-c)+b^r(b-c)(b-a)+c^r(c-a)(c-b)\geq 0.$$

From the Cauchy-Schwartz Inequality we have, for real vectors $\boldsymbol{a}=(a_1 \dots a_n)$ and $\boldsymbol{b}=(b_1 \dots b_n)$,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

We also have the following famous inequality from Jordan: if $0<|x|\leq \pi/2$, then

$$\frac{2}{\pi}\leq \frac{\sin x}{x}<1.$$

The known inequalities are numerous, and the list is growing. For a survey of inequalities, texts from Bullen [Bul93] as well as Beckenbach and Bellman [BB61] are excellent resources.

While the topic of inequalities spans many branches of Mathematics, we will focus on those related to complex analysis, geometry, and matrix theory. Chapter 1 is dedicated to trigonometric and hyperbolic inequalities. We start with a proof of the famous Triangle Inequality and then look to an incredible result of Katsuura [Kat11]. Katsuura began by providing a new proof of the following:

$$\theta < \frac{\sin\theta + \tan\theta}{2}$$

for $0 < \theta < \pi/2$. He then extended it to a string of seven inequalities (see Corollary 1.1.2). We then explore three alternate proofs of Price's Inequality [KO07], which states that for a, b, θ real numbers, $a \neq b$, $a, b \geq 0$ and positive integer n,

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^2 + b^2 - 2ab\cos\theta} \leq \left(\frac{a^n - b^n}{a - b}\right)^2.$$

Next, we discuss new complex hyperbolic inequalities, such as the following: if $z = x + iy$ is a complex number with $x \neq k\pi$ for any integer k and n is a positive integer, then

$$\left|\frac{\tanh(nz)}{\tanh z}\right| \leq |\coth x|$$

[KO07]. At the end of Chapter 1, we look at a few trigonometric and hyperbolic inequalities derived from infinite products.

We begin Chapter 2 on geometric inequalities with two proofs of the Arithmetic-Geometric Mean Inequality. There are many inequalities involving the sides, angles, and areas of triangles. Some were introduced by famous mathematicians such as Carlitz (who published more than 700 papers on a variety of subjects) and Srivastava (who published more than 1,000 papers). Many of the results are well known, but we will provide new proofs for them. For instance, Euler's Inequality states that the circumradius of a triangle is never less than twice the inradius. Also, the Erdös-Mordell inequality tells us that for any point P in a triangle,

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3),$$

where R_1, R_2, R_3 are the distances from P to the vertices of the triangle and r_1, r_2, r_3 are the distances to the sides [Niv81]. Finally, we dedicate a section to the results of Day [Day91]. He found that the area of the triangle formed by three tangent lines to an ellipse is strictly greater than half that of the triangle formed by joining their points of tangency; and for a hyperbola, the inequality is reversed.

In our final chapter, our focus shifts to univalent functions and matrices. In the study of univalent functions, one cannot overlook the very famous Bieberbach Conjecture. Consider functions f that are analytic (differentiable) and univalent (one-to-one) in the unit disk, with the properties that $f(0) = 0$ and $f'(0) = 1$. It is known that f has the power series expansion $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Bieberbach correctly believed that for any n, $|a_n| \leq n$. While not proven for sixty-eight years (by de Branges), this conjecture, as well as other properties of univalent functions, sparked a great deal of research. We will discuss the Area Theorem, Koebe's One-Quarter Theorem, Bieberbach's proof for $n = 2$, and the proof of Bieberbach's Conjecture for real coefficients. For our last topic, we look at Hadamard's Inequality for determinants. His inequality asserts that for any $n \times n$ matrix $A = [a_{ij}]$,

$$|\det A|^2 \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^2\right).$$

We consider two proofs of his inequality in the real case, and one that proves it true for any complex matrix.

There are numerous books, papers, and journals dedicated to the study of mathematical inequalities, so it is worth mentioning some additional references. For trigonometric inequalities, see Andreescu and Feng [AF05]. See Kazarinoff for more geometric inequalities [Kaz61]. And, another interesting and useful reference for analytic inequalities is a book by Mitrinović, which includes a large number of inequalities and in many cases their proofs [Mit70].

CHAPTER 1

TRIGONOMETRIC AND HYPERBOLIC INEQUALITIES

In this chapter, we will explore several trigonometric inequalities. They include inequalities with sine and tangent, complex hyperbolic functions, and Price's Inequality. The results are quite elegant, and their proofs give us great insight into the properties of trigonometric functions.

1.1 Inequalities involving the sine and tangent functions

Here we will derive a chain of inequalities related to those introduced in calculus texts. This chain arose from the "Hungarian Problem Book II," a collection of contest problems that posed the following to its readers in 1909: show that the radian measure of an acute angle is less than the arithmetic mean of its sine and tangent [Kur63]. In 2011, Katsuura not only provided a new proof of the statement, but also extended it to include several inequalities [Kat11].

Theorem 1.1.1. *Let* $0 < \theta < \pi/2$. *Then*

$$\theta < \frac{\sin\theta + \tan\theta}{2}.$$

Proof. Before we begin, we state some well-known trigonometric identities.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{1.1.1}$$

$$\cos 2\theta = 2\cos^2\theta - 1 \tag{1.1.2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{1.1.3}$$

First notice that

$$\begin{aligned}
&\sin\theta\tan\theta - \left[2\tan\left(\frac{\theta}{2}\right)\right]^2 \\
&\qquad = \frac{\sin^2\theta}{\cos\theta} - \frac{4\sin^2(\theta/2)}{\cos^2(\theta/2)} \\
&\qquad = \frac{\sin^2\theta}{\cos^2(\theta/2) - \sin^2(\theta/2)} - \frac{4\sin^2(\theta/2)}{\cos^2(\theta/2)} \text{ , by (1.1.1)} \\
&\qquad = \frac{4\sin^2(\theta/2)\cos^2(\theta/2)}{\cos^2(\theta/2) - \sin^2(\theta/2)} - \frac{4\sin^2(\theta/2)}{\cos^2(\theta/2)} \text{ , by (1.1.3)} \\
&\qquad = \frac{4\sin^2(\theta/2)\,\{\cos^4(\theta/2) - [\cos^2(\theta/2) - \sin^2(\theta/2)]\}}{\cos^2(\theta/2)\,[\cos^2(\theta/2) - \sin^2(\theta/2)]}. \qquad (1.1.4)
\end{aligned}$$

The expression in braces in the numerator can be simplified to

$$\begin{aligned}
&\cos^4\left(\frac{\theta}{2}\right) - \left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right] \\
&\qquad = \cos^4\left(\frac{\theta}{2}\right) - \cos\theta \text{ , by (1.1.1)} \\
&\qquad = \cos^4\left(\frac{\theta}{2}\right) - \left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right) \text{ , by (1.1.2)} \\
&\qquad = \left(\cos^2\left(\frac{\theta}{2}\right) - 1\right)^2 > 0.
\end{aligned}$$

So, the entire numerator of (1.1.4) is greater than 0. Since, by (1.1.1)

$$\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos\theta > 0$$

for $0 < \theta < \pi/2$, the entire denominator of (1.1.4) is positive as well. Hence,

$$\sin\theta\tan\theta - \left[2\tan\left(\frac{\theta}{2}\right)\right]^2 > 0.$$

So,

$$2\tan\left(\frac{\theta}{2}\right) < \sqrt{\sin\theta\tan\theta}\,. \tag{1.1.5}$$

Next we apply the Arithmetic-Geometric Mean Inequality, which will be proven in Chapter 2. The Arithmetic-Geometric Mean Inequality states that for any set of n nonnegative real numbers $a_1, a_2, \dots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}. \tag{1.1.6}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$. Applying this to the positive numbers $\sin\theta$ and $\tan\theta$ we obtain

$$\frac{\sin\theta + \tan\theta}{2} > \sqrt{\sin\theta\tan\theta}\,. \tag{1.1.7}$$

(Notice here that we have a strict inequality since $\sin\theta \neq \tan\theta$ for $0 < \theta < \pi/2$.)

Now, consider Figure 1.1 below.

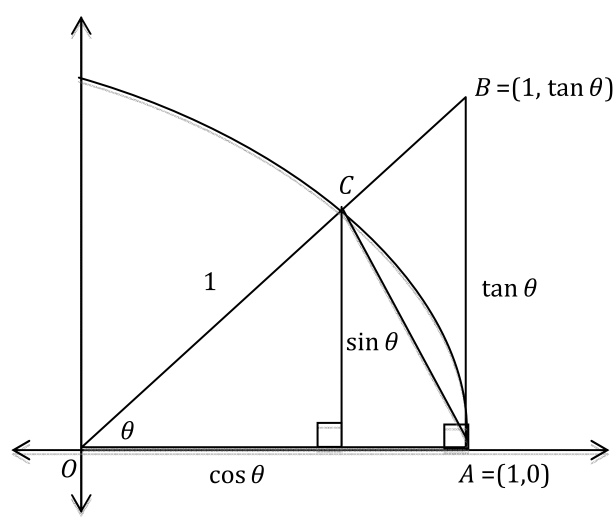

Figure 1.1: A sector of a circle with radius 1

From this figure, we have that

$$\Delta_{OAC} = \frac{1}{2}(1)(\sin\theta) = \frac{\sin\theta}{2}$$

and

$$\Delta_{OAB} = \frac{1}{2}(1)(\tan\theta) = \frac{\tan\theta}{2},$$

where Δ indicates the area of the triangle. Also,

$$\text{Area}(sector\ OAC) = \left(\frac{\theta}{2}\right)(1)^2 = \frac{\theta}{2}.$$

Because the area of triangle OAC is less than the area of sector OAC, which is less than the area of triangle OAB, we have that

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2},$$

or, equivalently,

$$\sin\theta < \theta < \tan\theta\ . \tag{1.1.8}$$

(Note here that it is a strict inequality since for $0 < \theta < \pi/2$ we have $\cos\theta < 1$.) And if we replace θ by $\theta/2$ in (1.1.8), we obtain:

$$2\sin\left(\frac{\theta}{2}\right) < \theta < 2\tan\left(\frac{\theta}{2}\right). \tag{1.1.9}$$

Finally, we combine (1.1.9), (1.1.5), and (1.1.7). This gives us:

$$2\sin\left(\frac{\theta}{2}\right) < \theta < 2\tan\left(\frac{\theta}{2}\right) < \sqrt{\sin\theta\tan\theta} < \frac{\sin\theta + \tan\theta}{2}.$$

Hence,

$$\theta < \frac{\sin\theta + \tan\theta}{2},$$

our desired result. ■

The following is an extension of Theorem 1.1.1.

Corollary 1.1.2. *Let* $0 < \theta < \pi/2$. *Then*

$$\sin\theta < 2\sin\left(\frac{\theta}{2}\right) < \theta < \sin\left(\frac{\theta}{2}\right) + \tan\left(\frac{\theta}{2}\right) < 2\tan\left(\frac{\theta}{2}\right)$$

$$< \sqrt{\sin\theta\tan\theta} < \frac{\sin\theta + \tan\theta}{2} < \tan\theta.$$

Proof. We replace θ by $\theta/2$ in Theorem 1.1.1., and we get that

$$\theta < \sin\left(\frac{\theta}{2}\right) + \tan\left(\frac{\theta}{2}\right).$$

Combining the above inequality with (1.1.9) yields

$$2\sin\left(\frac{\theta}{2}\right) < \theta < \sin\left(\frac{\theta}{2}\right) + \tan\left(\frac{\theta}{2}\right). \tag{1.1.10}$$

Notice that

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) < 2\sin\left(\frac{\theta}{2}\right), \tag{1.1.11}$$

from (1.1.3) and the fact that $\cos(\theta/2) < 1$ for $0 < \theta < \pi/2$. And so, by (1.1.11), (1.1.10), (1.1.8), (1.1.5), (1.1.7), and (1.1.8),

$$\sin\theta < 2\sin\left(\frac{\theta}{2}\right) < \theta < \sin\left(\frac{\theta}{2}\right) + \tan\left(\frac{\theta}{2}\right) < 2\tan\left(\frac{\theta}{2}\right)$$

$$< \sqrt{\sin\theta\tan\theta} < \frac{\sin\theta + \tan\theta}{2} < \tan\theta. \quad ■$$

1.2 Price's Inequality

In 2002, Thomas E. Price discovered the inequality below [Pri02].

Theorem 1.2.1. (Price's Inequality) *Let $a, b,$ and θ be real numbers with a and b non-negative and non-equal. Also, let n be an integer greater than or equal to 1. Then,*

$$\frac{a^{2n}+b^{2n}-2a^n b^n\cos(n\theta)}{a^2+b^2-2ab\cos\theta} \leq \left(\frac{a^n-b^n}{a-b}\right)^2.$$

Interestingly, the main objective of Price's paper was not to discover the inequality above; rather, he was working to generalize the formula for the products of chord lengths of a circle to that of the ellipse. Consider the unit circle with $n > 1$ equally spaced points on its circumference. Call these points $\omega_0, \omega_1, \ldots, \omega_{n-1}$, where ω_0 is the base point. Draw the chords connecting ω_0 to each of the other points. The product of the lengths of the n – 1 chords is n, which we will show using the process below.

We first choose $\omega_0 = 1$. Then, $\omega_j = e^{2i\pi j/n}$ for $j = 0, 1, \ldots, n-1$ are the n roots of unity, or equivalently, the roots of the polynomial $z^n - 1$. So,

$$z^n - 1 = (z-1)(z-\omega_1)(z-\omega_2)\ldots(z-\omega_{n-1}),$$

or,

$$\frac{z^n-1}{z-1} = (z-\omega_1)(z-\omega_2)\ldots(z-\omega_{n-1}).$$

Therefore,

$$\lim_{z\to 1}\left|\frac{z^n-1}{z-1}\right| = \lim_{z\to 1}|z-\omega_1||z-\omega_2|\ldots|z-\omega_{n-1}|$$

$$= |1 - \omega_1||1 - \omega_2| \dots |1 - \omega_{n-1}|$$

$$= \prod_{j=1}^{n-1} |1 - \omega_j|,$$

which is precisely the product of the lengths of the chords. So, the evaluation of this product is equivalent to calculating the limit above. Using L'Hôpital's rule (see Brown and Churchill [BC09]), we have

$$\lim_{z \to 1} \left| \frac{z^n - 1}{z - 1} \right| = \left| \lim_{z \to 1} \left(\frac{z^n - 1}{z - 1} \right) \right| = \left| \lim_{z \to 1} \left(\frac{n z^{n-1}}{1} \right) \right| = |n| = n.$$

Hence, the product of the lengths of the chords in any circle is *n.*

Price then applied a similar process to the ellipse. Let a, b, and θ be real numbers with a and b not equal and nonnegative, and $0 \leq \theta < 2\pi$. (If a were equal to b, we would have the special case of the circle.) For appropriate values for a and b, the following equation describes an ellipse with major axis vertices $\pm(a + b)$, minor axis vertices $\pm i(a - b)$, and foci equidistant from the origin and on the real axis:

$$a e^{i\theta} + b e^{-i\theta} = (a + b) \cos \theta + i(a - b) \sin \theta.$$

The chords of the ellipse are then constructed choosing the n points on the ellipse that are the images of the n roots of unity of the circle under the mapping $e^{i\theta} \to a e^{i\theta} + b e^{-i\theta}$ with $\omega_0 = a + b$. Then, the product of the lengths of the chords, d_n, of the ellipse, is given by:

$$d_n = n \frac{a^n - b^n}{a - b}.$$

Price achieved this result after a great deal of calculation. This led him to the inequality in Theorem 1.2.1.

We will now look at three alternate proofs of Price's Inequality found by Katsuura and Obaid [KO07], each of which is simpler and shorter than that of Price's. Each proof will be discussed in detail, as well as some additional inequalities established by Katsuura and Obaid. Before we do so, notice that Price's Inequality does not hold when we replace cosine by sine. For example, suppose $a = 2, b = 1, n = 2$, and $\theta = \pi/2$. Then,

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \sin(n\theta)}{a^2 + b^2 - 2ab\sin\theta} = \frac{16 + 1 - 8(0)}{4 + 1 - 2(2)(1)} = 17 > 9 = \left(\frac{4-1}{2-1}\right)^2 = \left(\frac{a^n - b^n}{a - b}\right)^2.$$

Our first proof of Price's Inequality uses some elementary inequalities. We begin by establishing the following lemmas.

Lemma 1.2.2. *Suppose n is a positive integer and θ is a real number. Then,*

$$\sin(n\theta) \cdot \sin\theta \leq 2n(1 - \cos\theta).$$

Proof. First suppose that $-1 \leq \cos\theta < 0$. Then $2n(1 - \cos\theta) \leq 2n(2) = 4n$. But, $\sin(n\theta) \cdot \sin\theta \leq 1$. So, our inequality is trivially true. Suppose then, $0 \leq \cos\theta \leq 1$. We will proceed with a proof by induction.

<u>Base Step</u>: Suppose $n = 1$. Then,

$$\begin{aligned} 2n(1 - \cos\theta) - \sin(n\theta) \cdot \sin\theta &= 2(1 - \cos\theta) - (1 - \cos^2\theta) \\ &= 2 - 2\cos\theta - 1 + \cos^2\theta \\ &= \cos^2\theta - 2\cos\theta + 1 \\ &= (\cos\theta - 1)^2 \geq 0. \end{aligned}$$

Induction Step. Suppose our inequality holds for a positive integer n. Then,

$\sin[(n+1)\theta] \cdot \sin\theta = \sin(n\theta + \theta) \cdot \sin\theta$

$$= [\sin(n\theta) \cdot \cos\theta + \cos(n\theta) \cdot \sin\theta][\sin\theta]$$

$$= \sin(n\theta) \cdot \sin\theta \cdot \cos\theta + \sin^2\theta \cdot \cos(n\theta)$$

$$\leq 2n(1-\cos\theta) \cdot \cos\theta + (1-\cos^2\theta) \cdot \cos(n\theta), \text{by the induction assumption}$$

$$= 2n(1-\cos\theta) \cdot \cos\theta + (1+\cos\theta)(1-\cos\theta) \cdot \cos(n\theta)$$

$$= [1-\cos\theta][2n\cos\theta + (1+\cos\theta) \cdot \cos(n\theta)]. \tag{1.2.1}$$

Because $0 \leq \cos\theta \leq 1$, we have

$$[1-\cos\theta][2n\cos\theta + (1+\cos\theta) \cdot \cos(n\theta)] \leq (1-\cos\theta)(2n+2)$$

$$\leq (1-\cos\theta)(2n+2)$$

$$= 2n + 2 - 2n\cos\theta - 2\cos\theta$$

$$= 2(n+1) - 2\,(n+1)\cos\theta$$

$$= 2(n+1)(1-\cos\theta). \tag{1.2.2}$$

And so, by (1.2.1) and (1.2.2),

$$\sin[(n+1)\theta] \cdot \sin\theta \leq 2(n+1)(1-\cos\theta),$$

completing our induction. ■

Lemma 1.2.3. *Suppose n is a positive integer and θ is a real number. Then,*

$$1 - \cos(n\theta) \leq n^2(1-\cos\theta).$$

Proof: Again, we will proceed with a proof by induction.

Base Step: Suppose $n = 1$. The inequality is trivially true.

Induction Step. Suppose our inequality holds for positive integer n. Then,

$$1 - \cos[(n+1)\theta] = 1 - \cos(n\theta + \theta)$$

$$= 1 - [\cos(n\theta) \cdot \cos\theta - \sin(n\theta) \cdot \sin\theta]$$

$$= 1 - \cos(n\theta) \cdot \cos\theta + \sin n\theta \cdot \sin\theta. \qquad (1.2.3)$$

By Lemma 1.2.2,

$$1 - \cos(n\theta) \cdot \cos\theta + \sin n\theta \cdot \sin\theta$$

$$\leq 1 - \cos(n\theta) \cdot \cos\theta + 2n(1 - \cos\theta)$$

$$= 1 - \cos(n\theta) \cdot \cos\theta + 2n(1 - \cos\theta) + \cos\theta - \cos\theta$$

$$= [1 - \cos(n\theta)] \cdot \cos\theta + (1 - \cos\theta) + 2n(1 - \cos\theta). \qquad (1.2.4)$$

Applying the induction hypothesis, we get,

$$[1 - \cos(n\theta)] \cdot \cos\theta + (1 - \cos\theta) + 2n(1 - \cos\theta)$$

$$\leq n^2(1 - \cos\theta) \cdot \cos\theta + (1 - \cos\theta) + 2n(1 - \cos\theta)$$

$$= (1 - \cos\theta)(n^2 \cos\theta + 2n + 1). \qquad (1.2.5)$$

Finally, since $\cos\theta \leq 1$,

$$(1 - \cos\theta)(n^2 \cos\theta + 2n + 1)$$

$$\leq (1 - \cos\theta)(n^2 + 2n + 1)$$

$$= (n+1)^2(1 - \cos\theta) \qquad (1.2.6)$$

And so, by (1.2.3) – (1.2.6),

$$1 - \cos[(n+1)\theta] \leq (n+1)^2(1 - \cos\theta),$$

which completes our induction and our proof. ∎

Lemma 1.2.4. *Suppose a and b are positive real numbers and n and k are positive integers. Then,*

$$\sum_{k=0}^{n} a^k b^{n-k} \geq (n+1)a^{n/2}b^{n/2}.$$

Proof. Let $x = a/b > 0$. Then,

$$\begin{aligned}
\sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{n} x^k b^n = b^n \sum_{k=0}^{n} x^k & \\
&= b^n(x^0 + x^1 + \cdots + x^n) \\
&= b^n(n+1)\left(\frac{x^0 + x^1 + \cdots + x^n}{n+1}\right) \\
&\geq b^n(n+1)(x^0 \cdot x^1 \cdot \ldots \cdot x^n)^{[1/(n+1)]} \\
&= b^n(n+1)(x^{0+1+\cdots+n})^{[1/(n+1)]}, \qquad (1.2.7)
\end{aligned}$$

with the inequality coming from the Arithmetic-Geometric Mean Inequality (1.1.6).

Now, applying the formula for the sum of a finite arithmetic sequence, we have that

$$\begin{aligned}
b^n(n+1)(x^{0+1+\cdots+n})^{[1/(n+1)]} &= b^n(n+1)x^{[(n+1)(0+n)/2]\cdot[1/(n+1)]} \\
&= b^n(n+1)x^{n/2} \\
&= b^n(n+1)\left(\frac{a}{b}\right)^{n/2} \\
&= (n+1)a^{n/2}b^{n/2}. \qquad (1.2.8)
\end{aligned}$$

Therefore, by (1.2.7) and (1.2.8),

$$\sum_{k=0}^{n} a^k b^{n-k} \geq (n+1)a^{n/2}b^{n/2},$$

as desired. ■

We are now ready to complete our first proof of Theorem 1.2.1, Price's Inequality.

Proof 1. Assume that a, b, and θ be real numbers with a and b nonnegative and non-equal. Also, let n be an integer greater than or equal to 1. Notice that if $\theta = 2k\pi$ for some integer k, then Theorem 1.2.1 is trivial. Assume then, that $\theta \neq 2k\pi$ for any integer k. Now, let

$$P = (a^n - b^n)^2(a^2 + b^2 - 2ab\cos\theta) - (a-b)^2[a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)]$$

Then, by adding and subtracting $2a^n b^n$ in the last set of parenthesis, we get that

$$\begin{aligned} P &= (a^n - b^n)^2(a^2 + b^2 - 2ab\cos\theta) \\ &\quad -(a-b)^2[a^{2n} - 2a^n b^n + b^{2n} + 2a^n b^n - 2a^n b^n \cos(n\theta)] \\ &= (a^n - b^n)^2(a^2 + b^2 - 2ab\cos\theta) \\ &\quad -(a-b)^2[(a^n - b^n)^2 + 2a^n b^n - 2a^n b^n \cos(n\theta)] \\ &= (a^n - b^n)^2(a^2 + b^2 - 2ab\cos\theta) \\ &\quad -(a^n - b^n)^2(a-b)^2 - 2a^n b^n[1 - \cos(n\theta)](a-b)^2 \\ &= (a^n - b^n)^2(a^2 + b^2 - 2ab\cos\theta) \\ &\quad -(a^n - b^n)^2(a^2 + b^2 - 2ab) - 2a^n b^n[1 - \cos(n\theta)](a-b)^2. \end{aligned}$$

And, with a great deal of algebra, we obtain,

$$P = 2ab(1-\cos\theta)\left\{(a^n - b^n)^2 - \left[\frac{1-\cos(n\theta)}{1-\cos\theta}\right] a^{n-1}b^{n-1}(a-b)^2\right\}$$

Applying Lemma 1.2.3,

$$\begin{aligned} P &\geq 2ab(1-\cos\theta)[(a^n - b^n)^2 - n^2 a^{n-1}b^{n-1}(a-b)^2] \\ &= 2ab(1-\cos\theta)(a-b)^2\left[\left(\frac{a^n - b^n}{a-b}\right)^2 - n^2 a^{n-1}b^{n-1}\right] \end{aligned}$$

$$= 2ab(1-\cos\theta)(a-b)^2\left\{\left[b^{n-1}\left(\frac{\frac{b^n-a^n}{b^n}}{\frac{b-a}{b}}\right)\right]^2 - n^2a^{n-1}b^{n-1}\right\}$$

$$= 2ab(1-\cos\theta)(a-b)^2\left(\left\{b^{n-1}\left[\frac{1-\left(\frac{a}{b}\right)^n}{1-\frac{a}{b}}\right]\right\}^2 - n^2a^{n-1}b^{n-1}\right) \qquad (1.2.9)$$

Now, the formula for the sum, S, of a finite geometric sequence with common ratio r and n terms is well known to be:

$$S = \frac{1-r^n}{1-r}. \qquad (1.2.10)$$

Applying (1.2.10) to (1.2.9), we get

$$P \geq 2ab(1-\cos\theta)(a-b)^2\left\{\left[b^{n-1}\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^k\right]^2 - n^2a^{n-1}b^{n-1}\right\}.$$

Since $b^{n-1}\sum_{k=0}^{n-1}(a/b)^k = \sum_{k=0}^{n-1}a^kb^{(n-1)-k}$, we can apply Lemma 1.2.4. So,

$$P \geq 2ab(1-\cos\theta)(a-b)^2\left[\left(na^{\frac{n-1}{2}}b^{\frac{n-1}{2}}\right)^2 - n^2a^{n-1}b^{n-1}\right]$$

$$= 2ab(1-\cos\theta)(a-b)^2(n^2a^{n-1}b^{n-1} - n^2a^{n-1}b^{n-1})$$

$$= 0.$$

Hence,

$$\frac{a^{2n}+b^{2n}-2a^nb^n\cos(n\theta)}{a^2+b^2-2ab\cos\theta} \leq \left(\frac{a^n-b^n}{a-b}\right)^2.$$

This completes our first proof of Price's Inequality. ■

Our next proof uses complex numbers, some properties of hyperbolic functions, and the following theorems and lemmas.

Theorem 1.2.5. (The Triangle Inequality) *For any two complex numbers* z_1 *and* z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof. Let z_1 and z_2 be complex numbers. Then,

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2 \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + 2Re(z_1\overline{z_2}) + |z_2|^2 \\
&\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2.
\end{aligned}$$

And when we take the square root of both sides, we have our desired inequality. ■

Lemma 1.2.6. *Suppose z is a complex number such that* $|z| \neq 1$ *and n is a positive integer. Then,*

$$\left|\frac{z^n - 1}{z - 1}\right| \leq \frac{|z|^n - 1}{|z| - 1}.$$

Proof. By (1.2.10) above, we have

$$\left|\frac{z^n - 1}{z - 1}\right| = \left|\sum_{k=0}^{n-1} z^k\right|. \tag{1.2.11}$$

Also, notice that

$$\left|\sum_{k=0}^{n-1} z^k\right| = |z^0 + z^1 + \cdots + z^{n-1}|$$

$$\leq |z^0| + |z^1| + \cdots + |z^{n-1}|, \tag{1.2.12}$$

where we used the Triangle Inequality. In addition,

$$|z^0| + |z^1| + \cdots + |z^{n-1}| = \sum_{k=0}^{n-1} |z^k| = \frac{|z|^n - 1}{|z| - 1}, \tag{1.2.13}$$

again by (1.2.10). So, (1.2.11) – (1.2.13) gives us

$$\left|\frac{z^n - 1}{z - 1}\right| \leq \frac{|z|^n - 1}{|z| - 1}. \quad \blacksquare$$

Theorem 1.2.7. *Let $z = x + iy$ be a complex number with x nonzero and suppose n is a positive integer. Then,*

$$\left|\frac{\sinh(nz)}{\sinh z}\right| \leq \frac{\sinh(nx)}{\sinh x}.$$

Proof. Suppose $z = x + iy$. Note that

$$|e^z| = \left|e^x \cdot e^{iy}\right| = |e^x| \cdot |\cos y + i \sin y| = e^x. \tag{1.2.14}$$

It is easy to see that

$$\left|\frac{\sinh(nz)}{\sinh z}\right| = \left|\frac{e^{nz} - e^{-nz}}{e^z - e^{-z}}\right|$$

$$= \left|\left(\frac{e^z \cdot e^{nz}}{e^z \cdot e^{nz}}\right)\left(\frac{e^{nz} - e^{-nz}}{e^z - e^{-z}}\right)\right|$$

$$= \left|\frac{e^z(e^{2nz} - 1)}{e^{nz}(e^{2z} - 1)}\right|$$

$$= \left|\frac{e^z}{e^{nz}}\right| \left|\frac{e^{2nz} - 1}{e^{2z} - 1}\right|$$

$$= \left(\frac{e^{-nx}}{e^{-x}}\right)\left|\frac{e^{2nz}-1}{e^{2z}-1}\right|, \tag{1.2.15}$$

by (1.2.14). Lemma 1.2.6 and (1.2.14) give

$$\left(\frac{e^{-nx}}{e^{-x}}\right)\left|\frac{e^{2nz}-1}{e^{2z}-1}\right| \le \frac{e^{-nx}}{e^{-x}}\left(\frac{|e^{2z}|^n-1}{|e^{2z}|-1}\right)$$

$$= \frac{e^{-nx}}{e^{-x}}\left(\frac{e^{2nx}-1}{e^{2x}-1}\right)$$

$$= \frac{e^{nx}-e^{-nx}}{e^{x}-e^{-x}}$$

$$= \frac{\sinh(nx)}{\sinh x}. \tag{1.2.16}$$

Hence, by (1.2.15) and (1.2.16), we obtain

$$\left|\frac{\sinh(nz)}{\sinh z}\right| \le \frac{\sinh(nx)}{\sinh x}. \quad \blacksquare$$

To simplify our second proof of Price's Inequality, we will list two useful properties of hyperbolic functions. Again, suppose $z = x + iy$ is a complex number.

$$|\sinh z|^2 = (\sinh z)\left(\overline{\sinh z}\right)$$

$$= (\sinh z)(\sinh \bar{z})$$

$$= \frac{\left(e^{z+\bar{z}} + e^{-(z+\bar{z})}\right) - \left(e^{z-\bar{z}} + e^{-(z-\bar{z})}\right)}{4}$$

$$= \frac{\left(e^{2x} + e^{-2x}\right) - \left(e^{2iy} + e^{-2iy}\right)}{4}$$

$$= \frac{\cosh(2x) - \cosh(2iy)}{2}$$

$$= \frac{\cosh(2x) - \cos(2y)}{2}. \tag{1.2.17}$$

Similarly, we have

$$|\cosh z|^2 = \frac{\cosh(2x) + \cos(2y)}{2} \tag{1.2.18}$$

Proof 2. Again assume a, b, and θ are real numbers with a and b positive and non-equal. (Notice that if $a = 0$ or $b = 0$, then Price's Inequality is trivially true.) Also, let n be an integer greater than or equal to 1. Let $c = a/b$ and $z = (\ln c + i\theta)/2$. Then, by (1.2.17),

$$\begin{aligned}
|\sinh(nz)|^2 &= \frac{\cosh(n \cdot \ln c) - \cos(n\theta)}{2} \\
&= \frac{e^{n \cdot \ln c} + e^{-n \cdot \ln c} - 2\cos(n\theta)}{4} \\
&= \frac{c^n + c^{-n} - 2\cos(n\theta)}{4} \\
&= \frac{(a^n/b^n) + (b^n/a^n) - 2\cos(n\theta)}{4} \\
&= \frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{4a^n b^n}. \qquad (1.2.19)
\end{aligned}$$

Therefore,

$$\left|\frac{\sinh(nz)}{\sinh z}\right|^2 = \frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{(ab)^{n-1}(a^2 + b^2 - 2ab\cos\theta)}.$$

And so,

$$\begin{aligned}
\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^2 + b^2 - 2ab\cos\theta} &= (ab)^{n-1} \left|\frac{\sinh(nz)}{\sinh z}\right|^2 \\
&\le (ab)^{n-1} \left[\frac{\sinh\left(\frac{n \cdot \ln c}{2}\right)}{\sinh\left(\frac{\ln c}{2}\right)}\right]^2,
\end{aligned}$$

by Theorem 1.2.7. Finally then,

$$\frac{a^{2n}+b^{2n}-2a^nb^n\cos(n\theta)}{a^2+b^2-2ab\cos\theta} \leq (ab)^{n-1}\left[\frac{\cosh(n\cdot\ln c)-\cos(0)}{\cosh(\ln c)-\cos(0)}\right], \text{by } (1.2.17)$$

$$= (ab)^{n-1}\left[\frac{\dfrac{e^{n\cdot\ln c}+e^{-n\cdot\ln c}-2}{2}}{\dfrac{e^{\ln c}+e^{-\ln c}-2}{2}}\right]$$

$$= (ab)^{n-1}\left(\frac{c^n+c^{-n}-2}{c+c^{-1}-2}\right)$$

$$= (ab)^{n-1}\left(\frac{(a/b)^n+(a/b)^{-n}-2}{(a/b)+(a/b)^{-1}-2}\right)$$

$$= (ab)^{n-1}\left(\frac{\dfrac{a^{2n}+b^{2n}-2a^nb^n}{a^nb^n}}{\dfrac{a^2+b^2-2ab}{ab}}\right)$$

$$= (ab)^{n-1}\left(\frac{a^{2n}+b^{2n}-2a^nb^n}{a^2+b^2-2ab}\right)\left(\frac{ab}{a^nb^n}\right)$$

$$= \left(\frac{a^{2n}+b^{2n}-2a^nb^n}{a^2+b^2-2ab}\right)$$

$$= \left(\frac{a^n-b^n}{a-b}\right)^2,$$

as desired. ■

Our third proof of Price's Inequality is the easiest of the four known proofs. It comes from simply applying Lemma 1.2.6.

Proof 3. Again assume a, b, and θ are real numbers with a and b positive and non-equal. Also, let n be an integer greater than or equal to 1. By squaring both sides of the equation in Lemma 1.2.6, we get that

$$\left|\frac{z^n-1}{z-1}\right|^2 \le \left(\frac{|z|^n-1}{|z|-1}\right)^2. \tag{1.2.20}$$

Also, notice that

$$\begin{aligned}
|z^n-1|^2 = (z^n-1)(\overline{z^n-1}) &= (z^n-1)(\overline{z^n}-1) \\
&= |z^n|^2 - z^n - \overline{z^n} + 1 \\
&= |z^n|^2 - 2Re(z^n) + 1. \qquad (1.2.21)
\end{aligned}$$

Now, we let $z=(b/a)e^{i\theta}$. Then, by (1.2.20) and (1.2.21) we obtain

$$\frac{\left|\frac{b^n}{a^n}e^{in\theta}\right|^2 - 2Re\left(\frac{b^n}{a^n}e^{in\theta}\right)+1}{\left|\frac{b}{a}e^{i\theta}\right|^2 - 2Re\left(\frac{b}{a}e^{i\theta}\right)+1} \le \left(\frac{\left|\frac{b}{a}e^{i\theta}\right|^n-1}{\left|\frac{b}{a}e^{i\theta}\right|-1}\right)^2. \tag{1.2.22}$$

But,

$$\begin{aligned}
\frac{\left|\frac{b^n}{a^n}e^{in\theta}\right|^2 - 2Re\left(\frac{b^n}{a^n}e^{in\theta}\right)+1}{\left|\frac{b}{a}e^{i\theta}\right|^2 - 2Re\left(\frac{b}{a}e^{i\theta}\right)+1} &= \frac{\frac{b^{2n}}{a^{2n}} - 2\frac{b^n}{a^n}\cos(n\theta)+1}{\frac{b^2}{a^2} - 2\frac{b}{a}\cos\theta + 1} \\
&= \frac{\frac{a^{2n}+b^{2n}-2a^nb^n\cos(n\theta)}{a^{2n}}}{\frac{a^2+b^2-2ab\cos\theta}{a^2}}.
\end{aligned}$$

So, by (1.2.22),

$$\frac{\frac{a^{2n}+b^{2n}-2a^nb^n\cos(n\theta)}{a^{2n}}}{\frac{a^2+b^2-2ab\cos\theta}{a^2}} \le \left(\frac{\left|\frac{b}{a}e^{i\theta}\right|^n-1}{\left|\frac{b}{a}e^{i\theta}\right|-1}\right)^2.$$

Thus,

$$\frac{a^{2n}+b^{2n}-2a^nb^n\cos(n\theta)}{a^2+b^2-2ab\cos\theta} \le \left(\frac{\left|\frac{b}{a}e^{i\theta}\right|^n-1}{\left|\frac{b}{a}e^{i\theta}\right|-1}\right)^2 \left(\frac{a^2}{a^{2n}}\right)$$

$$= \left(\frac{\frac{b^n}{a^n} - 1}{\frac{b}{a} - 1}\right)^2 \left(\frac{a^2}{a^{2n}}\right)$$

$$= \left(\frac{\frac{b^n - a^n}{a^n}}{\frac{b - a}{a}}\right)^2 \left(\frac{a^2}{a^{2n}}\right)$$

$$= \left(\frac{b^n - a^n}{b - a}\right)^2 \left(\frac{a^{2n}}{a^2}\right)\left(\frac{a^2}{a^{2n}}\right)$$

$$= \left(\frac{a^n - b^n}{a - b}\right)^2,$$

as desired. Hence, Price's Inequality is just a special case of Lemma 1.2.6 ■

1.3 Inequalities similar to Price's Inequality

Interestingly, we can develop new inequalities similar to Price's, using similar processes. Two of these are discussed below [KO07].

Theorem 1.3.1. *Let a, b, and θ be real numbers with a and b non-equal and nonnegative. Also, let n be an integer greater than or equal to 1. Then,*

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)} \leq \left(\frac{a + b}{a - b}\right)^2 \left(\frac{a^2 + b^2 - 2ab\cos\theta}{a^2 + b^2 + 2ab\cos\theta}\right).$$

Proof. As we did in our second proof of Price's Inequality, we let $c = a/b$ and $z = x + iy = (\ln c + i\theta)/2$. Applying (1.2.18), we obtain that

$$|\cosh(nz)|^2 = \frac{\cosh(n\ln c) + \cos(n\theta)}{2}$$

$$= \frac{e^{n\ln c} + e^{-n\ln c}}{4} + \frac{\cos(n\theta)}{2}$$

$$= \frac{c^n + c^{-n} + 2\cos(n\theta)}{4}$$

$$= \frac{(a^n/b^n) + (b^n/a^n) + 2\cos(n\theta)}{4}$$

$$= \frac{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}{4a^n b^n}. \qquad (1.3.1)$$

So, by (1.3.1) and (1.2.19),

$$|\tanh(nz)|^2 = \frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}. \qquad (1.3.2)$$

and

$$|\coth(nz)|^2 = \frac{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}.$$

Therefore,

$$|\coth x|^2 = \frac{a^2 + b^2 + 2ab\cos(0)}{a^2 + b^2 - 2ab\cos(0)}$$

$$= \frac{a^2 + b^2 + 2ab}{a^2 + b^2 - 2ab}$$

$$= \left(\frac{a+b}{a-b}\right)^2. \qquad (1.3.3)$$

In the next section (Theorem 1.4.3), we will prove that if $z = x + iy$ is a complex number with $x \neq k\pi$ for any integer k and n is a positive integer, then

$$\left|\frac{\tanh(nz)}{\tanh z}\right| \leq |\coth x|.$$

So,

$$\left|\frac{\tanh(nz)}{\tanh z}\right|^2 \leq |\coth x|^2,$$

which gives us that

$$|\tanh(nz)|^2 \le |\coth x|^2 |\tanh z|^2. \qquad (1.3.4)$$

Hence, (1.3.2) – (1.3.4) give us the following inequality, very similar to Price's:

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(n\theta)}{a^{2n} + b^{2n} + 2a^n b^n \cos(n\theta)} \le \left(\frac{a+b}{a-b}\right)^2 \left(\frac{a^2 + b^2 - 2ab\cos\theta}{a^2 + b^2 + 2ab\cos\theta}\right). \quad \blacksquare$$

Interestingly, the right-hand side of the above inequality is independent of n.

Theorem 1.3.2. *Let θ, a, and b be positive real numbers with a not equal to b and $n = 4m + 1$, for some integer m. Then*

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \sin(n\theta)}{a^2 + b^2 - 2ab\sin\theta} \le \left(\frac{a^n - b^n}{a-b}\right)^2.$$

Proof. Again, let $c = a/b$ and $z = (\ln c + i\theta)/2$. Then,

$$\begin{aligned} nz - i\left(\frac{\pi}{4}\right) &= n\left(z - i\frac{\pi}{4n}\right) \\ &= n\left(\frac{\ln c + i\theta}{2} - i\frac{\pi}{4n}\right) \\ &= n\left[\frac{\ln c}{2} + i\left(\frac{\theta}{2} - \frac{\pi}{4n}\right)\right]. \end{aligned} \qquad (1.3.5)$$

So, by (1.2.19) together with (1.3.5), we have that

$$\begin{aligned} \left|\sinh\left[nz - i\left(\frac{\pi}{4}\right)\right]\right|^2 &= \left|\sinh\left\{n\left[\frac{\ln c}{2} + i\left(\frac{\theta}{2} - \frac{\pi}{4n}\right)\right]\right\}\right|^2 \\ &= \frac{\left\{a^{2n} + b^{2n} - 2a^n b^n \cos\left[2n\left(\frac{\theta}{2} - \frac{\pi}{4n}\right)\right]\right\}}{4a^n b^n} \\ &= \frac{\left[a^{2n} + b^{2n} - 2a^n b^n \cos\left(n\theta - \frac{\pi}{2}\right)\right]}{4a^n b^n} \end{aligned}$$

$$= \frac{[a^{2n} + b^{2n} - 2a^n b^n \sin(n\theta)]}{4a^n b^n}.$$

And so,

$$a^{2n} + b^{2n} - 2a^n b^n \sin(n\theta) = 4(ab)^n \cdot \left|\sinh\left[nz - i\left(\frac{\pi}{4}\right)\right]\right|^2.$$

Therefore,

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \sin(n\theta)}{a^2 + b^2 - 2ab\sin(n\theta)} = \frac{4(ab)^n \cdot \left|\sinh\left[nz - i\left(\frac{\pi}{4}\right)\right]\right|^2}{4ab \cdot \left|\sinh\left[z - i\left(\frac{\pi}{4}\right)\right]\right|^2}$$

$$= (ab)^{n-1}\left|\frac{\sinh\left(nz - i\frac{\pi}{4}\right)}{\sinh\left(z - i\frac{\pi}{4}\right)}\right|^2. \qquad (1.3.6)$$

But,

$$\left|\frac{\sinh\left(nz - i\frac{\pi}{4}\right)}{\sinh\left(z - i\frac{\pi}{4}\right)}\right|^2 = \left|\frac{e^{nz - i\frac{\pi}{4}} - e^{-nz + i\frac{\pi}{4}}}{e^{z - i\frac{\pi}{4}} - e^{-z + i\frac{\pi}{4}}}\right|^2$$

$$= \left|\left(\frac{e^{i\frac{\pi}{4}}}{e^{i\frac{\pi}{4}}}\right)\left(\frac{e^{nz - i\frac{\pi}{4}} - e^{-nz + i\frac{\pi}{4}}}{e^{z - i\frac{\pi}{4}} - e^{-z + i\frac{\pi}{4}}}\right)\right|^2$$

$$= \left|\frac{e^{nz} - e^{-nz}e^{i\frac{\pi}{2}}}{e^{z} - e^{-z}e^{i\frac{\pi}{2}}}\right|^2$$

$$= \left|\frac{e^{nz} - ie^{-nz}}{e^{z} - ie^{-z}}\right|^2$$

$$= \left|\left(\frac{-ie^{z}e^{nz}}{-ie^{z}e^{nz}}\right)\left(\frac{e^{nz} - ie^{-nz}}{e^{z} - ie^{-z}}\right)\right|^2$$

$$= \left|\left(\frac{e^{z}}{e^{nz}}\right)\left(\frac{-ie^{2nz} - 1}{-ie^{2z} - 1}\right)\right|^2$$

$$= \left(\frac{e^{2x}}{e^{2nx}}\right)\left|\frac{-ie^{2nz}-1}{-ie^{2z}-1}\right|^2, \text{ by (1.2.14)}$$

$$= \left(e^{-2x(n-1)}\right)\left|\frac{-ie^{2nz}-1}{-ie^{2z}-1}\right|^2. \qquad (1.3.7)$$

So, (1.3.6) and (1.3.7) give us

$$\frac{a^{2n}+b^{2n}-2a^nb^n\sin(n\theta)}{a^2+b^2-2ab\sin(n\theta)} = (ab)^{n-1}\left(e^{-2x(n-1)}\right)\left|\frac{-ie^{2nz}-1}{-ie^{2z}-1}\right|^2. \qquad (1.3.8)$$

Next, we let $t = -ie^{2z}$. Notice that with $n = 4m+1$, we have

$$t^n = (-ie^{2z})^n = (-i)^{4m+1}e^{2nz} = -ie^{2nz}.$$

And so, with Lemma (1.2.6),

$$\left(e^{-2x(n-1)}\right)\left|\frac{-ie^{2nz}-1}{-ie^{2z}-1}\right|^2 = \left(e^{-2x(n-1)}\right)\left|\frac{t^n-1}{t-1}\right|^2$$

$$\leq \left(e^{-2x(n-1)}\right)\left(\frac{|t|^n-1}{|t|-1}\right)^2$$

$$= \left(e^{-2x(n-1)}\right)\left(\frac{|-ie^{2z}|^n-1}{|-ie^{2z}|-1}\right)^2$$

$$= \left(e^{-2x(n-1)}\right)\left(\frac{e^{2nx}-1}{e^{2x}-1}\right)^2, \text{ by (1.2.14)}$$

$$= \left[\left(e^{-x(n-1)}\right)\left(\frac{e^{2nx}-1}{e^{2x}-1}\right)\right]^2$$

$$= \left[\left(\frac{e^{-nx}}{e^{-x}}\right)\left(\frac{e^{2nx}-1}{e^{2x}-1}\right)\right]^2$$

$$= \left[\frac{e^{nx}-e^{-nx}}{e^x-e^{-x}}\right]^2$$

$$= \left[\frac{e^{\frac{n\ln c}{2}} - e^{-\frac{n\ln c}{2}}}{e^{\frac{\ln c}{2}} - e^{-\frac{\ln c}{2}}}\right]^2$$

$$= \left[\frac{c^{\frac{n}{2}} - c^{-\frac{n}{2}}}{c^{\frac{1}{2}} - c^{-\frac{1}{2}}}\right]^2$$

$$= \left[\frac{\left(\frac{a}{b}\right)^{\frac{n}{2}} - \left(\frac{a}{b}\right)^{-\frac{n}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}} - \left(\frac{a}{b}\right)^{-\frac{1}{2}}}\right]^2. \tag{1.3.9}$$

Hence, by (1.3.8) and (1.3.9),

$$\frac{a^{2n} + b^{2n} - 2a^n b^n \sin(n\theta)}{a^2 + b^2 - 2ab\sin(n\theta)} \leq (ab)^{n-1}\left[\frac{\left(\frac{a}{b}\right)^{\frac{n}{2}} - \left(\frac{a}{b}\right)^{-\frac{n}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}} - \left(\frac{a}{b}\right)^{-\frac{1}{2}}}\right]^2$$

$$= \left[(ab)^{\frac{n-1}{2}}\left(\frac{\left(\frac{a}{b}\right)^{\frac{n}{2}} - \left(\frac{a}{b}\right)^{-\frac{n}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}} - \left(\frac{a}{b}\right)^{-\frac{1}{2}}}\right)\right]^2$$

$$= \left[\frac{(ab)^{\frac{n}{2}}}{(ab)^{\frac{1}{2}}}\left(\frac{\left(\frac{a}{b}\right)^{\frac{n}{2}} - \left(\frac{a}{b}\right)^{-\frac{n}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}} - \left(\frac{a}{b}\right)^{-\frac{1}{2}}}\right)\right]^2$$

$$= \left(\frac{a^n - b^n}{a - b}\right)^2. \quad \blacksquare$$

1.4 Complex hyperbolic inequalities

In addition to exploring additional proofs of Price's Inequality, Katsuura and Obaid developed several hyperbolic inequalities from the properties they used in these proofs. With some manipulation, we obtain the following as a result of Theorem 1.2.7 [KO07].

Corollary 1.4.1. *Suppose $z = x + iy$ is a complex number.*

(a) If $x \neq 0$, then

$$\left|\frac{\sin(inz)}{\sin(iz)}\right| \leq \frac{\sin(inx)}{\sin(ix)}.$$

(b) If $y \neq 0$, then

$$\left|\frac{\sin(nz)}{\sin z}\right| \leq \frac{\sinh(ny)}{\sinh y}.$$

Proof. Because $\sinh z = -i\sin(iz)$, we obtain that

$$\left|\frac{\sinh(nz)}{\sinh(z)}\right| = \left|\frac{-i\sin(inz)}{-i\sin(iz)}\right| = \left|\frac{\sin(inz)}{\sin(iz)}\right|$$

and

$$\frac{\sinh(nx)}{\sinh(x)} = \frac{-i\sin(inx)}{-i\sin(ix)} = \frac{\sin(inx)}{\sin(ix)},$$

where $x \neq 0$. Therefore, by Theorem 1.2.7 and the two equations above,

$$\left|\frac{\sin(inz)}{\sin(iz)}\right| \leq \frac{\sin(inx)}{\sin(ix)},$$

which is (a).

Now, suppose that $y \neq 0$, and replace z by iz in part (a) above. Then, for the left-hand side, we have

$$\left|\frac{\sin(-nz)}{\sin(-z)}\right| = \left|\frac{\sin(nz)}{\sin z}\right|.$$

And for the right-hand side of (a) we have,

$$\frac{\sin(-iny)}{\sin(-iy)} = \frac{-i\sin(iny)}{-i\sin(iy)} = \frac{\sinh(ny)}{\sinh y}.$$

Hence, by part (a) and the two equations above,

$$\left|\frac{\sin(nz)}{\sin z}\right| \le \frac{\sinh(ny)}{\sinh y},$$

which is (b). ■

Corollary 1.4.2. *Let $z = x + iy$ be a complex number with x nonzero and let n be a positive integer. Then*

$$\frac{\cosh(2nx) - \cos(2ny)}{\cosh(2x) - \cos(2y)} \le \left(\frac{\sinh(nx)}{\sinh x}\right)^2.$$

Proof. From (1.2.17),

$$\left|\frac{\sinh(nz)}{\sinh z}\right|^2 = \frac{\cosh(2nx) - \cos(2ny)}{\cosh(2x) - \cos(2y)}.$$

Now, if we square both sides of Theorem 1.2.7, we have

$$\left|\frac{\sinh(nz)}{\sinh z}\right|^2 \le \left(\frac{\sinh(nx)}{\sinh x}\right)^2.$$

Hence,

$$\frac{\cosh(2nx) - \cos(2ny)}{\cosh(2x) - \cos(2y)} \le \left(\frac{\sinh(nx)}{\sinh x}\right)^2. \quad ■$$

Theorem 1.4.3. *Let $z = x + iy$ be a complex number with $x \neq k\pi$ for any integer k. Also, let n be a positive integer. Then*

$$\left|\frac{\tanh(nz)}{\tanh z}\right| \le |\coth x|.$$

Proof. First notice that

$$\left|\frac{\cosh z}{\cosh(nz)}\right| = \left|\frac{e^{z}+e^{-z}}{e^{nz}+e^{-nz}}\right| = \left|\left(\frac{e^{z}}{e^{z}}\right)\left(\frac{e^{nz}}{e^{nz}}\right)\left(\frac{e^{z}+e^{-z}}{e^{nz}+e^{-nz}}\right)\right|$$

$$= \left|\frac{e^{nz}}{e^{z}}\right|\left|\frac{e^{2z}+1}{e^{2nz}+1}\right|$$

$$= \left(\frac{e^{nx}}{e^{x}}\right)\left|\frac{e^{2z}+1}{e^{2nz}+1}\right|, \tag{1.4.1}$$

by (1.2.14). Now, by the Triangle Inequality [Bul98],

$$\left(\frac{e^{nx}}{e^{x}}\right)\left|\frac{e^{2z}+1}{e^{2nz}+1}\right| \le \left(\frac{e^{nx}}{e^{x}}\right)\left(\frac{|e^{2z}|+|1|}{\left||e^{2nz}|-|1|\right|}\right)$$

$$= \left(\frac{e^{-x}}{e^{-nx}}\right)\left(\frac{e^{2x}+1}{|e^{2nx}-1|}\right)$$

$$= \frac{e^{x}+e^{-x}}{|e^{nx}-e^{-nx}|}$$

$$= \frac{\cosh x}{|\sinh(nx)|}. \tag{1.4.2}$$

Hence, by (1.4.1) and (1.4.2),

$$\left|\frac{\cosh z}{\cosh(nz)}\right| \le \frac{\cosh x}{|\sinh(nx)|}.$$

Therefore,

$$\left|\frac{\tanh(nz)}{\tanh z}\right| = \left|\frac{\sinh(nz)}{\sinh z}\right|\left|\frac{\cosh z}{\cosh(nz)}\right|$$

$$\le \left|\frac{\sinh(nz)}{\sinh z}\right|\left(\frac{\cosh x}{|\sinh(nx)|}\right)$$

$$\le \left(\frac{\sinh(nx)}{\sinh x}\right)\left(\frac{\cosh x}{|\sinh(nx)|}\right) \text{, by Theorem 1.2.7}$$

$$= \left(\frac{\sinh(nx)}{|\sinh(nx)|}\right)\left(\frac{\cosh x}{\sinh x}\right)$$

$$= (\pm 1)(\coth x)$$

$$= |\coth x|. \quad \blacksquare$$

Interestingly, the right-hand side of the inequality above is independent of both n and y.

Corollary 1.4.4. *Let $z = x + iy$ be a complex number with $y \neq k\pi$ for any integer k. Also, let n be a positive integer. Then*

$$\left|\frac{\tan(nz)}{\tan z}\right| \leq |\coth y|.$$

Proof. First note that $\tanh z = i\tan(iz)$. Then,

$$\left|\frac{\tanh\,(nz)}{\tanh\,(z)}\right| = \left|\frac{i\tan(inz)}{i\tan(iz)}\right| = \left|\frac{\tan(inz)}{\tan(iz)}\right|.$$

So, by Theorem 1.4.3, we have the following (when $x \neq 0$):

$$\left|\frac{\tan(inz)}{\tan(iz)}\right| \leq |\coth x|. \tag{1.4.3}$$

Replace z by iz in (1.4.3). Then, for the left-hand side we have that

$$\left|\frac{\tan(-nz)}{\tan(-z)}\right| = \left|\frac{\tan(nz)}{\tan z}\right|.$$

And for the right-hand side of (1.4.3) we have

$$|\coth(-y)| = \left|\frac{\cosh(-y)}{\sinh(-y)}\right| = \left|\frac{\cosh y}{-\sinh y}\right| = \left|\frac{\cosh y}{\sinh y}\right| = |\coth y|.$$

Hence, by (1.4.3) and the two equations above,

$$\left|\frac{\tan(nz)}{\tan z}\right| \leq |\coth y|. \quad \blacksquare$$

1.5 Trigonometric and hyperbolic inequalities from infinite products

Jordan and Kober's inequalities are two well-known trigonometric results, each with numerous applications and appearances in mathematical works [CZQ]. Jordan's inequality states that:

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1, \quad 0 < x \leq \frac{\pi}{2}.$$

Kober's is the following:

$$\cos x \geq 1 - \frac{2x}{\pi}, \quad 0 \leq x \leq \frac{\pi}{2}.$$

Interestingly, his inequality reverses for $x \in (\pi/2, \pi)$. There are many extensions of these inequalities as well as variations of them. For instance, in 1969, R. Redheffer found that, for all real values of x,

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

In this section, we will explore inequalities appearing in the work of Chen, Zhao, and Qi [CZQ]. These will include inequalities of the cosine function, the hyperbolic cosine function, and the hyperbolic sine function, all of which are similar to Redheffer's above. We will utilize mathematical induction as well as the infinite product forms of each of these functions.

Theorem 1.5.1: *If* $-1/2 < x < 1/2$*, then*

$$\cos(\pi x) \geq \frac{1 - 4x^2}{1 + 4x^2}.$$

Proof: Because $\cos x = \cos(-x)$, it is sufficient to prove the inequality for values between 0 and $1/2$. Notice that if $x = 0$ then we have equality. It is known that [DR03]

$$\cos(\pi x) = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2n-1)^2}\right)$$

$$= (1 - 4x^2)\prod_{n=2}^{\infty}\left(1 - \frac{4x^2}{(2n-1)^2}\right).$$

We let

$$F_n = \prod_{k=2}^{n}\left(1 - \frac{4x^2}{(2k-1)^2}\right),$$

for $n = 2, 3, \ldots$ Then,

$$F_{n+1} = F_n\left(1 - \frac{4x^2}{(2n+1)^2}\right). \tag{1.5.1}$$

So,

$$\cos(\pi x) = (1 - 4x^2)\left[\lim_{n\to\infty} F_n\right]$$

$$= \frac{(1 - 4x^2)}{(1 + 4x^2)}\left[(1 + 4x^2)\lim_{n\to\infty} F_n\right]. \tag{1.5.2}$$

Using these facts and mathematical induction, we will prove that, for $n = 2, 3, \ldots$

$$(1 + 4x^2)F_n > 1 + \frac{4x^2}{2n-1}. \tag{1.5.3}$$

<u>Base Step:</u> Suppose $n = 2$. Then,

$$(1 + 4x^2)F_n - 1 - \frac{4x^2}{2n-1} = (1 + 4x^2)F_2 - 1 - \frac{4x^2}{2(2)-1}$$

$$= (1+4x^2)\left(1-\frac{4x^2}{9}\right)-1-\frac{4x^2}{3}$$

$$= 1-\frac{4x^2}{9}+4x^2-\frac{16x^4}{9}-1-\frac{4x^2}{3}$$

$$= \frac{20x^2}{9}-\frac{16x^4}{9}$$

$$= \frac{4x^2}{9}(5-4x^2) > 0,$$

since $0 < x < 1/2$. Hence, for $n = 2$, inequality (1.5.3) holds.

Induction Step: Suppose (1.5.3) is true for some $n \geq 2$. Then,

$$(1+4x^2)F_{n+1}-1-\frac{4x^2}{2(n+1)-1}$$

$$= (1+4x^2)F_n\left(1-\frac{4x^2}{(2n+1)^2}\right)-1-\frac{4x^2}{2n+1} \text{ , by (1.5.1)}$$

$$> \left(1+\frac{4x^2}{2n-1}\right)\left(1-\frac{4x^2}{(2n+1)^2}\right)-1-\frac{4x^2}{2n+1} \text{ , by induction assumption}$$

$$= 1-\frac{4x^2}{(2n+1)^2}+\frac{4x^2}{2n-1}-\frac{16x^4}{(2n-1)(2n+1)^2}-1-\frac{4x^2}{2n+1}$$

$$= 4x^2\left[\frac{-1}{(2n+1)^2}+\frac{1}{2n-1}-\frac{4x^2}{(2n-1)(2n+1)^2}-\frac{1}{2n+1}\right]$$

$$= 4x^2\left[\frac{-(2n-1)+(2n+1)^2-4x^2-(2n-1)(2n+1)}{(2n-1)(2n+1)^2}\right]$$

$$= 4x^2\,\frac{-2n+1+4n^2+4n+1-4x^2-4n^2+1)}{(2n-1)(2n+1)^2}$$

$$= 4x^2\left[\frac{3+2n-4x^2}{(2n-1)(2n+1)^2}\right].$$

Now, clearly $4x^2 \geq 0$ and $(2n+1)^2 \geq 0$. Since $n \geq 2$ and $0 < x < 1/2$, we also have $2n - 1 > 0$ and

$$3 + 2n - 4x^2 \geq 3 + 2(2) - 4x^2 \geq 7 - 4x^2 > 7 - 4(0)^2 > 0.$$

Therefore,

$$4x^2\left(\frac{3 + 2n - 4x^2}{(2n-1)(2n+1)^2}\right) > 0,$$

and so,

$$(1 + 4x^2)F_{n+1} > 1 + \frac{4x^2}{2(n+1) - 1},$$

which completes our induction and proves (1.5.3).

Next, we evaluate the limit as n approaches infinity for each side of (1.5.3) and get

$$\lim_{n\to\infty}(1 + 4x^2)F_n \geq \lim_{n\to\infty}\left[1 + \frac{4x^2}{2n-1}\right] = 1 + 0 = 1. \qquad (1.5.4)$$

Notice that the above inequality is not strict. This is because of the known fact that if $a_n > b_n$ then

$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n. \qquad (1.5.5)$$

Clearly (1.5.2) and (1.5.4) yield

$$\cos(\pi x) = \frac{(1 - 4x^2)}{(1 + 4x^2)}\left[(1 + 4x^2)\lim_{n\to\infty} F_n\right] \geq \frac{(1 - 4x^2)}{(1 + 4x^2)},$$

as desired. ■

Using a process like that above, we can also prove the following similar theorem. Therefore, we leave the proof of Theorem 1.5.2 to the reader, and move on to Theorem 1.5.3.

Theorem 1.5.2. *If* $-1/2 < x < 1/2$ *, then*

$$\cosh(\pi x) \leq \frac{1+4x^2}{1-4x^2}.$$

Theorem 1.5.3. *If* $0 < |x| < 1$*, then*

$$\frac{\sinh(\pi x)}{\pi x} \leq \frac{1+x^2}{1-x^2}.$$

Proof: Notice first that the left-hand side of the inequality above is always positive. The right-hand side is negative if $|x| > 1$. Therefore, these are not possible values for x, and so it must be the case that $0 < |x| < 1$. It is known that [DR03]

$$\frac{\sinh(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right)$$

$$= (1+x^2)\prod_{n=2}^{\infty}\left(1+\frac{x^2}{n^2}\right).$$

Let

$$G_n = \prod_{k=2}^{n}\left(1+\frac{x^2}{k^2}\right)$$

for $n = 2, 3, \ldots$ Then,

$$G_{n+1} = G_n\left(1+\frac{x^2}{(n+1)^2}\right). \tag{1.5.6}$$

So,

$$\frac{\sinh(\pi x)}{\pi x} = (1+x^2)\left[\lim_{n\to\infty} G_n\right]$$

$$= \frac{(1+x^2)}{(1-x^2)}\left[(1-x^2)\lim_{n\to\infty} G_n\right]. \tag{1.5.7}$$

Using the equations above and mathematical induction again, we will prove that, for $n = 2, 3, \ldots$

$$(1-x^2)G_n < 1 - \frac{x^2}{n}. \tag{1.5.8}$$

<u>Base Step:</u> Suppose $n = 2$. Then,

$$(1-x^2)G_n - 1 + \frac{x^2}{n} = (1-x^2)G_2 - 1 + \frac{x^2}{2}$$

$$= (1-x^2)\left(1 + \frac{x^2}{(2)^2}\right) - 1 + \frac{x^2}{2}$$

$$= (1-x^2)\left(1 + \frac{x^2}{4}\right) - 1 + \frac{x^2}{2}$$

$$= 1 + \frac{x^2}{4} - x^2 - \frac{x^4}{4} - 1 + \frac{x^2}{2}$$

$$= -\frac{x^2}{4} - \frac{x^4}{4}$$

$$= -\left(\frac{x^2}{4} + \frac{x^4}{4}\right) < 0.$$

Hence, for $n = 2$, inequality (1.5.8) holds.

<u>Induction Step:</u> Suppose (1.5.8) is true for some $n \geq 2$. Thus we have

$$(1-x^2)G_{n+1} - 1 + \frac{x^2}{n+1}$$

$$= (1-x^2)G_n\left[1+\frac{x^2}{(n+1)^2}\right]-1+\frac{x^2}{n+1} \text{ , by (1.5.6)}$$

$$< \left(1-\frac{x^2}{n}\right)\left[1+\frac{x^2}{(n+1)^2}\right]-1+\frac{x^2}{n+1} \text{ , by induction assumption}$$

$$= 1+\frac{x^2}{(n+1)^2}-\frac{x^2}{n}-\frac{x^4}{n(n+1)^2}-1+\frac{x^2}{n+1}$$

$$= \frac{nx^2-(n+1)^2x^2-x^4+n(n+1)x^2}{n(n+1)^2}$$

$$= \frac{nx^2-n^2x^2-2nx^2-x^2-x^4+n^2x^2+nx^2}{n(n+1)^2}$$

$$= \frac{-x^2-x^4}{n(n+1)^2}$$

$$= -\left[\frac{x^2+x^4}{n(n+1)^2}\right] < 0.$$

which completes our induction and proved (1.5.8).

Next, we evaluate the limit as n approaches infinity for each side of (1.5.8) and have

$$\lim_{n\to\infty}(1-x^2)G_n \leq \lim_{n\to\infty}\left[1-\frac{x^2}{n}\right] = 1-0 = 1. \tag{1.5.9}$$

Again, the inequality above is not strict by (1.5.5). Combining (1.5.7) and (1.5.9) yields

$$\frac{\sinh(\pi x)}{\pi x} = \frac{(1+x^2)}{(1-x^2)}\left[(1-x^2)\lim_{n\to\infty} G_n\right] \leq \frac{1+x^2}{1-x^2},$$

as desired. ■

CHAPTER 2

GEOMETRIC INEQUALITIES

In this chapter, we examine geometric inequalities related to the triangle and conic sections. While there are numerous inequalities about the parts of the triangle, we mostly focus on those that have quite simple results but are in fact very applicable and quite interesting. Following this discussion about triangle inequalities, we consider inequalities that arise from conic sections and their tangent lines.

2.1 The Arithmetic-Geometric Mean Inequality

In the next section, we develop several inequalities about the angles, sides, and altitudes of a triangle. Interestingly, many of them are found by simply applying the Arithmetic-Geometric Mean Inequality. There are quite a few proofs of this famous inequality, including Cauchy's lengthy algebraic proof by induction, but the two we look at are much shorter and very elegant. The first was completed by George Pólya [HLP64] and the second is an application of the Lagrange method [Ste12].

Theorem 2.1.1. (The Arithmetic-Geometric Mean Inequality) *For any set of n nonnegative real numbers,* $a_1, a_2, \dots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof 1. Let $a_1, a_2, \ldots, a_n$ be a list of n nonnegative real numbers. We also let A denote their arithmetic mean and G their geometric mean. That is,

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n} \text{ and } G = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

We start by showing that $f(x) = e^x - x - 1 \geq 0$ for all real values of x. Well, $f'(x) = e^x - 1 = 0$ when $x = 0$. So, $x = 0$ is our only critical value for $f(x)$. Since $f''(x) = e^x$ is not only continuous near $x = 0$ but also positive at $x = 0$, we can apply the Second Derivative Test (see Stewart [Ste12]). The Second Derivative Test tells us that $f(x)$ has a local minimum at $x = 0$ and hence an absolute minimum since this is our only critical value. Therefore, $f(0) = 0$ is the absolute minimum value, implying that $f(x) = e^x - x - 1 \geq 0$, or $e^x \geq 1 + x$, for all real values of x. If we let $x = (a_i/A) - 1$, then we have

$$e^{\frac{a_i}{A}-1} \geq 1 + \frac{a_i}{A} - 1 = \frac{a_i}{A},$$

for all $i = 1, 2, \ldots, n$. So,

$$\left(e^{\frac{a_1}{A}-1}\right)\left(e^{\frac{a_2}{A}-1}\right)\ldots\left(e^{\frac{a_n}{A}-1}\right) \geq \left(\frac{a_1}{A}\right)\left(\frac{a_2}{A}\right)\ldots\left(\frac{a_n}{A}\right),$$

which gives us

$$e^{\frac{a_1+a_2+\cdots+a_n}{A}-n} \geq \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{A^n}.$$

But, with A and G as defined above, our inequality becomes

$$e^{n-n} = e^0 = 1 \geq \frac{G^n}{A^n}.$$

Hence,

$$A^n \geq G^n,$$

and therefore,

$$A \geq G. \quad \blacksquare$$

Proof 2: Suppose $a_1, a_2, \dots, a_n$are a list of *n* nonnegative real numbers. Let

$$f(a_1, a_2, \dots, a_n) = \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n},$$

with the constraint that these numbers add to a constant *c*. Now, we let

$$g(a_1, a_2, \dots, a_n) = a_1 + a_2 + \cdots + a_n = c.$$

We want to find the maximum value of *f* for which the constraint $g - c = 0$.

Consider

$$F(a_1, a_2, \dots, a_n, \lambda) = f(a_1, a_2, \dots, a_n) + \lambda[g(a_1, a_2, \dots, a_n) - c]$$
$$= \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n} + \lambda[a_1 + a_2 + \cdots + a_n - c],$$

where λ is a parameter called the Lagrange multiplier. Lagrange's method states that the maximum and/or minimum will occur where the partial derivatives of *F* (with respect to each variable) equal zero. So, we are looking to solve the following equations simultaneously:

$$\frac{\partial F}{\partial \lambda} = 0 \text{ and } \frac{\partial F}{\partial a_i} = 0$$

for each $1 \leq i \leq n$. Notice that

$$\frac{\partial f}{\partial a_1} = \sqrt[n]{a_2 \cdot \dots \cdot a_n} \cdot \frac{1}{n} \cdot a_1^{\frac{1}{n}-1} = \frac{1}{na_1} \sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n}$$

and

$$\frac{\partial g}{\partial a_1} = 1.$$

We do the same for each a_i and obtain

$$\frac{\partial F}{\partial a_1} = \frac{1}{na_1}\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} + \lambda(1) = 0,$$

$$\frac{\partial F}{\partial a_2} = \frac{1}{na_2}\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} + \lambda(1) = 0,$$

$$\vdots$$

$$\frac{\partial F}{\partial a_n} = \frac{1}{na_n}\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} + \lambda(1) = 0.$$

The equations above may be rewritten as

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = -\lambda n a_1 ,$$

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = -\lambda n a_2 ,$$

$$\vdots$$

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = -\lambda n a_n ,$$

and so it is clear that we must have

$$\lambda n a_1 = \lambda n a_2 = \cdots = \lambda n a_n.$$

Hence,

$$a_1 = a_2 = \cdots = a_n.$$

This, together with

$$\frac{\partial F}{\partial \lambda} = a_1 + a_2 + \cdots + a_n - c = 0.$$

yields

$$a_1 + a_2 + \cdots + a_n = na_i = c,$$

or,

$$a_i = \frac{c}{n}.$$

for each $1 \leq i \leq n$. Therefore,

$$\begin{aligned} f(a_1, a_2, \ldots, a_n) &= f\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right) \\ &= \sqrt[n]{\left(\frac{c}{n}\right)\left(\frac{c}{n}\right)\ldots\left(\frac{c}{n}\right)} \\ &= \sqrt[n]{\frac{c^n}{n^n}} \\ &= \frac{c}{n} \\ &= \frac{a_1 + a_2 + \cdots + a_n}{n}. \end{aligned}$$

Again, the Lagrange method guarantees that this is a maximum or minimum value for *f*. But, regardless of how small each a_i, there is no minimum value. Hence, this must be a maximum. Therefore,

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}. \quad \blacksquare$$

2.2 Inequalities in the triangle

Now that we have established the Arithmetic-Geometric Mean Inequality, we will define some notation that will be used throughout this chapter. Let *a*, *b*, and *c* be the lengths of the sides of ΔABC opposite the vertices *A*, *B*, and *C*, respectively. We let *r* be the radius of the inscribed circle (inradius) and *R* the radius of the circumscribed circle (circumradius). Finally, s will denote the semi-perimeter of ΔABC and Δ will be its area.

Throughout the following section, we also utilize the well-known Heron's Formula, which was discovered first by Heron of Alexandria. Below is an impressive proof of the formula that is not only simpler than Heron's, but was discovered by a high school student in 2007 [Edw07].

Theorem 2.2.1. (Heron's Formula) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Proof. Consider ΔABC in Figure 2.1.

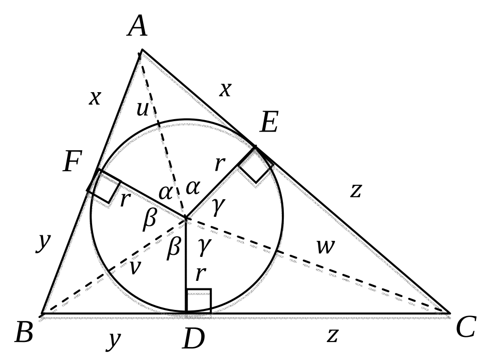

Figure 2.1: ΔABC with inscribed circle of radius r

We start by letting $a = y + z$, $b = x + z$, and $c = x + y$. Then $s = x + y + z$ will be the semi-perimeter. Clearly, $2\alpha + 2\beta + 2\gamma = 2\pi$. So, $\alpha + \beta + \gamma = \pi$. If we rotate our axis, then we have $r + ix = ue^{i\alpha}$, $r + iy = ve^{i\beta}$, and $r + iz = we^{i\gamma}$. So,

$$(r+ix)(r+iy)(r+iz) = \left(ue^{i\alpha}\right)\left(ve^{i\beta}\right)\left(we^{i\gamma}\right)$$

$$= uvwe^{i(\alpha+\beta+\gamma)} = uvwe^{i\pi} = -uvw.$$

Therefore,

$$\mathrm{Im}[(r+ix)(r+iy)(r+iz)] = r^2(x+y+z) - xyz = 0.$$

So,

$$r = \sqrt{\frac{xyz}{x+y+z}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence,

$$\begin{aligned}\Delta &= \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} \\ &= r\left(\frac{a+b+c}{2}\right) \\ &= rs \\ &= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \quad \blacksquare\end{aligned}$$

Now we establish the following theorem and two lemmas that will be used in the proof of Euler's Inequality [Kla67].

Lemma 2.2.2. $\Delta = rs$.

Proof. This was shown in the last few lines of Heron's Formula.

Lemma 2.2.3. $4R\Delta = abc$.

Proof. Consider ΔABC as in Figure 2.2 on the following page.

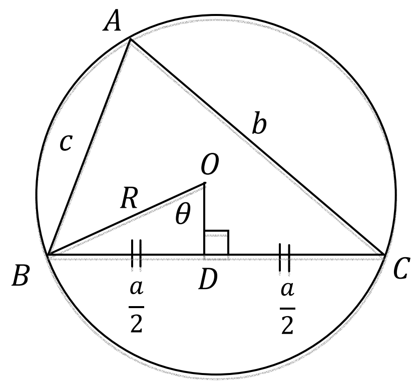

Figure 2.2: ΔABC inscribed in a circle of radius R

Notice that $m\angle A = (1/2)m\widehat{BC}$ and $m\angle\theta = (1/2)m\widehat{BC}$, so $\angle A \cong \angle\theta$. Since

$$\sin\theta = \frac{a}{2R},$$

we have

$$\sin A = \frac{a}{2R}, \tag{2.2.1}$$

or

$$R = \frac{a}{2\sin A} = \frac{abc}{2bc\sin A} \; .$$

Finally, since the area of ΔABC is

$$\Delta = \frac{bc\sin A}{2},$$

then

$$R = \frac{abc}{4\Delta},$$

as desired. ■

Theorem 2.2.4. $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$.

Proof. Suppose ΔABC is given as in Figure 2.1. Then,

$$a = y + z = r\left[\cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)\right]$$

$$= r\left[\frac{\cos(B/2)}{\sin(B/2)} + \frac{\cos(C/2)}{\sin(C/2)}\right]$$

$$= r\left[\frac{\sin(C/2)\cos(B/2) + \sin(B/2)\cos(C/2)}{\sin(B/2)\sin(C/2)}\right]$$

$$= r\left\{\frac{\sin[(B/2) + (C/2)]}{\sin(B/2)\sin(C/2)}\right\}, \tag{2.2.2}$$

Also, $m\angle A + m\angle B + m\angle C = 180$, and so $[(m\angle A)/2] + [(m\angle B)/2] + [(m\angle C)/2] =$ 90. Therefore, $[(B/2) + (C/2)]$ and $(A/2)$ are complementary, which gives us

$$\sin\left(\frac{B}{2} + \frac{C}{2}\right) = \cos\left(\frac{A}{2}\right).$$

With this and (2.2.2) we obtain

$$a = r\left[\frac{\cos(A/2)}{\sin(B/2)\sin(C/2)}\right],$$

and so

$$r = a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sec\left(\frac{A}{2}\right).$$

Next, we eliminate a from the equation above by means of (2.2.1) and get

$$r = 2R\sin A\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sec\left(\frac{A}{2}\right). \tag{2.2.3}$$

Now, note the following trigonometric identity:

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right),$$

or, equivalently,

$$\sin\theta \sec\left(\frac{\theta}{2}\right) = 2\sin\left(\frac{\theta}{2}\right). \tag{2.2.4}$$

So, by (2.2.3) and (2.2.4),

$$r = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right). \quad \blacksquare$$

There are many proofs in various mathematical publications of the following elegant result, which was first proved by Euler. Goldner completed his proof in 1950 [Gol50], and Klamkin did his in 1967 [Kla67].

Theorem 2.2.5. (Euler's Inequality) $R \geq 2r$.

Proof 1. [Gol50] Consider ΔABC with semi-perimeter s and side lengths of a, b, and c across from A, B, and C, respectively. Applying the Arithmetic-Geometric Mean Inequality (Theorem 2.1.1) to positive real numbers $s - b$ and $s - c$, we have

$$(s-b) + (s-c) \geq 2\sqrt{(s-b)(s-c)}.$$

But,

$$(s-b) + (s-c) = \frac{a+b+c}{2} - b + \frac{a+b+c}{2} - c = a.$$

So,

$$a \geq 2\sqrt{(s-b)(s-c)}.$$

Similarly,

$$b \geq 2\sqrt{(s-a)(s-c)}$$

and

$$c \geq 2\sqrt{(s-a)(s-b)}\,. \tag{2.2.5}$$

We multiply the left- and right-hand sides of these inequalities and get

$$abc \geq 8(s-a)(s-b)(s-c).$$

Now, since $4R\Delta = abc$, by Lemma 2.2.3, we have that

$$4R\Delta \geq 8(s-a)(s-b)(s-c).$$

Applying Heron's Formula (Theorem 2.2.1) to the inequality above, we obtain

$$4R\Delta \geq 8\frac{\Delta^2}{s}\,.$$

Lemma 2.2.2 states that $\Delta = rs$. So,

$$4R\Delta \geq 8r\Delta.$$

And finally,

$$R \geq 2r. \quad \blacksquare$$

Proof 2. [Kla67] Consider ΔABC as in Figure 2.3 below, with sides of length a, b, and c, across from vertices A, B, and C, respectively.

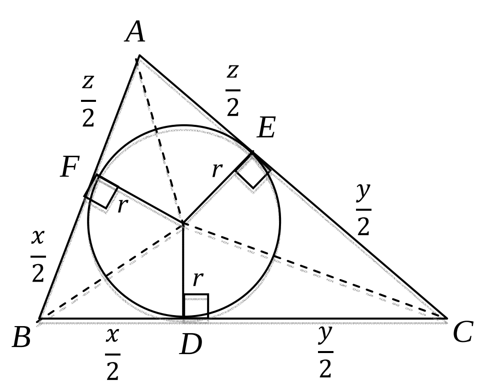

Figure 2.3: ΔABC with inradius r

We again begin by applying the Arithmetic-Geometric Mean Inequality (Theorem 2.1.1) three times, to the pairs (x, y), (y, z), and (z, x). Then,

$$x + y \geq 2\sqrt{xy}, \qquad y + z \geq 2\sqrt{yz}, \qquad z + x \geq 2\sqrt{zx}.$$

Upon multiplying, we obtain

$$(x + y)(y + z)(z + x) \geq 8xyz, \tag{2.2.6}$$

with equality when ΔABC is equilateral. Now, from Figure 2.3, we see that

$$x = 2(s - b), \qquad y = 2(s - c), \qquad z = 2(s - a). \tag{2.2.7}$$

So, if we replace x, y, and z in both sides of (2.2.6) as in (2.2.7), we get

$$8(2s - b - c)(2s - c - a)(2s - a - b) = 8abc \geq 8[8(s - b)(s - c)(s - a)].$$

Thus,

$$abc \geq 8(s - a)(s - b)(s - c).$$

Equivalently,

$$abcs \geq 8s(s - a)(s - b)(s - c).$$

Now, we apply Lemma 2.2.3 ($4R\Delta = abc$) to the left-hand side and Heron's Formula (Theorem 2.2.1) to the right-hand side and get that

$$4R\Delta s \geq 8\Delta^2.$$

So,

$$Rs \geq 2\Delta.$$

Then, by Lemma 2.2.2 ($\Delta = rs$),

$$Rs \geq 2rs.$$

Hence,

$$R \geq 2r. \quad \blacksquare$$

Our next Theorem, which relates the side lengths of a triangle to its circumradius, also comes from Goldner [Gol50].

Theorem 2.2.6. $(1/a) + (1/b) + (1/c) \geq [3/(2R)]$.

Proof. First, note that from (2.2.1), we have $R = [a/(2\sin A)]$. Similarly then,

$$R = \frac{c}{2\sin C}\ ,$$

or,

$$c = 2R\sin C.$$

This, together with (2.2.5), gives us that that

$$2R\sin C \geq 2\sqrt{(s-a)(s-b)}.$$

Equivalently,

$$\begin{aligned} 2R\sin C\ \sqrt{s(s-c)} &\geq 2\sqrt{s(s-a)(s-b)(s-c)} \\ &= 2\Delta, \quad \text{by Heron's Formula (Theorem 2.2.1)} \\ &= ab\sin C, \end{aligned}$$

since the area of any triangle is known to be

$$\Delta = \frac{ab\sin C}{2} = \frac{bc\sin A}{2} = \frac{ac\sin B}{2}. \tag{2.2.8}$$

Hence,

$$2R\sqrt{s(s-c)} \geq ab. \tag{2.2.9}$$

By the Arithmetic-Geometric Mean Inequality (Theorem 2.1.1),

$$\frac{s+(s-c)}{2} \geq \sqrt{s(s-c)} \tag{2.2.10}$$

Using (2.2.9) in (2.2.10) we have

$$\frac{s + (s - c)}{2} \geq \frac{ab}{2R} ,$$

or,

$$R(2s - c) \geq ab.$$

Thus,

$$R(a + b) \geq ab,$$

and so,

$$\frac{1}{a} + \frac{1}{b} \geq \frac{1}{R}.$$

Similarly,

$$\frac{1}{b} + \frac{1}{c} \geq \frac{1}{R}$$

and

$$\frac{1}{a} + \frac{1}{c} \geq \frac{1}{R}.$$

Adding these three, we obtain:

$$\frac{2}{a} + \frac{2}{b} + \frac{2}{c} \geq \frac{3}{R}.$$

Hence,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3}{2R} . \quad \blacksquare$$

We will continue with more inequalities regarding the triangle, with two theorems proven by Melville in 2004 [Mel04].

Theorem 2.2.7. $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta$.

Proof: Again, we consider ΔABC with side lengths of a, b, and c. Our proof begins with some simple algebra and applying a formula for the area of a triangle:

$$a^2+b^2+c^2-4\sqrt{3}\Delta = 2b^2+2c^2+a^2-b^2-c^2-4\sqrt{3}\left(\frac{1}{2}bc\sin A\right).$$

Now, since $a^2-b^2-c^2=-2bc\cos A$ (Law of Cosines), we have

$$a^2+b^2+c^2-4\sqrt{3}\Delta = 2b^2+2c^2-2bc\cos A-4\sqrt{3}\left(\frac{1}{2}bc\sin A\right)$$

$$= 2\left[b^2+c^2-bc\left(\cos A+\sqrt{3}\sin A\right)\right]. \qquad (2.2.11)$$

Using the difference identity for cosine, we have

$$\cos\left(A-\frac{\pi}{3}\right) = \cos A\cos\left(\frac{\pi}{3}\right)+\sin A\sin\left(\frac{\pi}{3}\right).$$

$$= \frac{1}{2}\cos A+\frac{\sqrt{3}}{2}\sin A.$$

Hence,

$$\cos A+\sqrt{3}\sin A = 2\cos\left(A-\frac{\pi}{3}\right).$$

And so, (2.2.11) becomes

$$a^2+b^2+c^2-4\sqrt{3}\Delta = 2\left[b^2+c^2-2bc\cos\left(A-\frac{\pi}{3}\right)\right]$$

$$\geq 2[b^2+c^2-2bc], \text{ since } -1\leq\cos(A-\pi/3)\leq 1$$

$$= 2(b-c)^2\geq 0.$$

Hence, $a^2+b^2+c^2-4\sqrt{3}\Delta\geq 0$, with equality when $A=\pi/3$ and $b = c$, i.e. when ΔABC is equilateral. This completes our proof. ■

Theorem 2.2.8. *Let h_A, h_B, and h_C denote the lengths of the altitudes drawn from A, B, and C, respectively. Then,*

$$\frac{1}{h_A^2} + \frac{1}{h_B^2} + \frac{1}{h_C^2} \geq \frac{\sqrt{3}}{\Delta}.$$

Proof. Using simple trigonometry, we can see that $h_A = b \sin C$ and $h_B = a \sin C$. So,

$$\frac{1}{h_A^2} + \frac{1}{h_B^2} = \frac{1}{b^2 \sin^2 C} + \frac{1}{a^2 \sin^2 C}$$

$$= \frac{a^2 + b^2}{a^2 b^2 \sin^2 C}$$

$$= \frac{a^2 + b^2}{4\Delta^2}, \tag{2.2.12}$$

by (2.2.8). Similarly,

$$\frac{1}{h_B^2} + \frac{1}{h_C^2} = \frac{b^2 + c^2}{4\Delta^2}, \tag{2.2.13}$$

and

$$\frac{1}{h_C^2} + \frac{1}{h_A^2} = \frac{c^2 + a^2}{4\Delta^2}. \tag{2.2.14}$$

Adding (2.2.12), (2.2.13), and (2.2.14) gives us

$$2\left(\frac{1}{h_A^2} + \frac{1}{h_B^2} + \frac{1}{h_C^2}\right) = \frac{2(a^2 + b^2 + c^2)}{4\Delta^2},$$

which simplifies to be

$$\frac{1}{h_A^2} + \frac{1}{h_B^2} + \frac{1}{h_C^2} = \frac{a^2 + b^2 + c^2}{4\Delta^2}.$$

Finally, we apply Theorem 2.2.7 and obtain our desired result:

$$\frac{1}{h_A^2} + \frac{1}{h_B^2} + \frac{1}{h_C^2} \geq \frac{\sqrt{3}}{\Delta}. \quad \blacksquare$$

Interestingly, upon combining Theorems 2.2.7 and 2.2.8 we get the following:

$$(a^2+b^2+c^2)\left(\frac{1}{h_A^2}+\frac{1}{h_B^2}+\frac{1}{h_C^2}\right)\geq 12.$$

Our next discoveries begin with an acute triangle, ΔABC, as pictured in Figure 2.4 below.

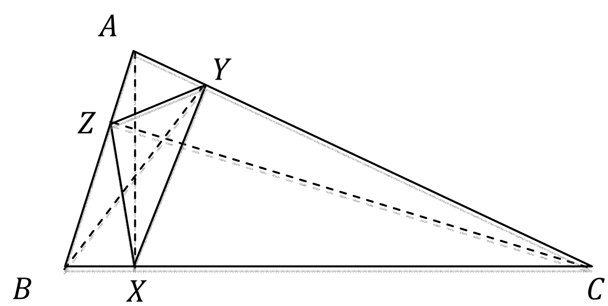

Figure 2.4: Acute ΔABC with inscribed pedal triangle, ΔXYZ

Suppose we draw the altitudes from A, B, and C to the sides opposite (the dashed lines). Call the points of intersection of the altitudes with the side of the triangle points X, Y, and Z, respectively. This triangle, ΔXYZ, is called the *pedal triangle*. The pedal triangle of an acute triangle has a perimeter no greater than any of the inscribed triangles. Garfunkel used this property to prove the theorems that follow [Gar69].

Theorem 2.2.9. *Suppose ΔABC is acute. Then,* $(abc)/(2R^2)\leq s$.

Proof: Consider acute ΔABC and pedal ΔXYZ as in Figure 2.4. Let P denote the perimeter of ΔXYZ. It is known that the perimeter of the pedal triangle of an acute triangle equals twice the area of the given triangle, divided by the radius of the circumscribed circle. Hence,

$$P = \frac{2\Delta}{R}$$

$$= \frac{2[(abc)/(4R)]}{R}, \qquad (2.2.15)$$

by Lemma 2.2.3. So,

$$P = \frac{abc}{2R^2}. \qquad (2.2.16)$$

Now, let a, b, and c be the lengths of the sides opposite A, B, and C, respectively. Consider inscribed $\Delta M_1M_2M_3$ formed by joining the midpoints of a, b, and c, as shown in Figure 2.5 below.

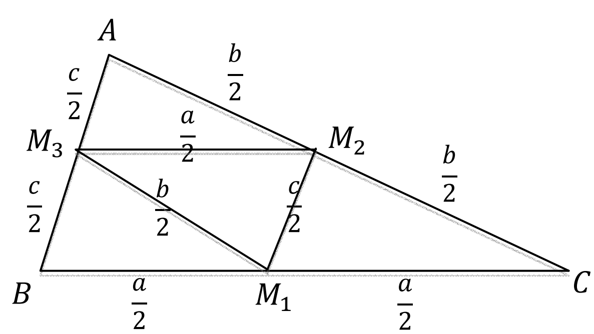

Figure 2.5: Acute ΔABC with midpoints of sides joined to form $\Delta M_1M_2M_3$

Then, $M_1M_2 = c/2$, $M_2M_3 = a/2$, and $M_3M_1 = b/2$. Let P' denote the perimeter of $\Delta M_1M_2M_3$. So,

$$P' = \frac{a+b+c}{2} = s. \qquad (2.2.17)$$

Now, using the fact that $P \leq P'$ (as discussed above) along with (2.2.16) and (2.2.17), we obtain our desired result:

$$\frac{abc}{2R^2} \leq s\,. \quad \blacksquare$$

Theorem 2.2.10. *Suppose* ΔABC *is acute. Then,*

$$\sin A + \sin B + \sin C \leq \cos(A/2) + \cos(B/2) + \cos(C/2).$$

Proof. Consider the circle with center I inscribed in ΔABC. Call the points of tangency D, E, and F (see Figure 2.6 below). Joining these points we get inscribed ΔDEF.

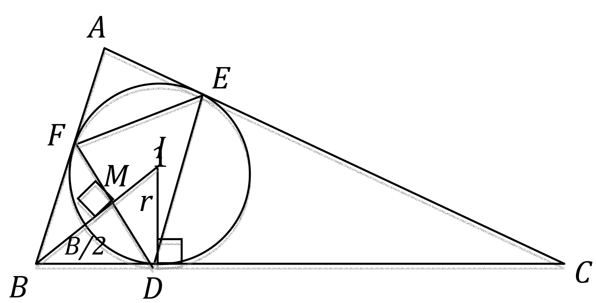

Figure 2.6: Acute ΔABC with inscribed circle

Now, let M be the midpoint of FD. Then, $m\angle BMF = m\angle DMB = 90°$. So,

$$\sin\left(\frac{B}{2}\right) = \frac{MD}{BD}$$

or,

$$MD = BD \cdot \sin\left(\frac{B}{2}\right). \tag{2.2.18}$$

Also, $m\angle IDC = m\angle IDB = 90°$ since r is a radius and D is a point of tangency. So, in ΔBID,

$$\tan\left(\frac{B}{2}\right) = \frac{r}{BD},$$

or,

$$BD = r \cdot \cot\left(\frac{B}{2}\right). \tag{2.2.19}$$

Combining (2.2.18) and (2.2.19) we have

$$MD = r \cot\left(\frac{B}{2}\right) \sin\left(\frac{B}{2}\right) = r \cos\left(\frac{B}{2}\right).$$

Hence, $DF = 2r\cos(B/2)$. Similarly, $DE = 2r\cos(C/2)$ and $EF = 2r\cos(A/2)$. Let P'' denote the perimeter of ΔDEF. Then,

$$P'' = 2r\left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right]. \tag{2.2.20}$$

Now, as stated above, the perimeter of the pedal triangle is no greater than the perimeter of any inscribed triangle. This, together with (2.2.16) and (2.2.20) gives us that

$$\frac{abc}{2R^2} \leq 2r\left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right]. \tag{2.2.21}$$

Now, by Lemmas 2.2.2 and 2.2.3,

$$abc = 4Rrs. \tag{2.2.22}$$

Assume, without loss of generality, that ΔABC is inscribed in a circle with a diagonal of length one (since we can always shrink a triangle down, preserving angle measure.) That is, $R = 1/2$. Then, (2.2.22) reduces to

$$abc = 2rs.$$

And so, by (2.2.21) and the equation above,

$$\frac{2rs}{2R^2} \leq 2r\left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right],$$

giving us

$$\frac{s}{2R^2} \leq \left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right],$$

which implies

$$\frac{a+b+c}{4R^2} \leq \left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right],$$

and so

$$a+b+c \leq \left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right], \tag{2.2.23}$$

since $R = 1/2$. Now, it is known that $(a/\sin A) = (b/\sin B) = (c/\sin C) = 2R$. With $R = 1/2$, then $a = \sin A$, $b = \sin B$, and $c = \sin C$. Using this fact in (2.2.23), we get our desired result:

$$\sin A + \sin B + \sin C \leq \cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right). \quad \blacksquare$$

2.3 The Erdös-Mordell Inequality

Our next geometric inequality was proposed by Paul Erdös in 1935 and later proven by L. J. Mordell in 1937 [Niv81].

Theorem 2.3.1. (Erdös-Mordell Inequality) *Suppose P is any point on the interior of ΔABC. Let R_1, R_2, and R_3 be the distances from P to the vertices of the triangle and r_1, r_2, and r_3 the distances from P to the sides. Then,*

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3),$$

with equality if and only if the triangle is equilateral and P is the centroid.

Proof. Let ΔABC be labeled as in Figure 2.7 below.

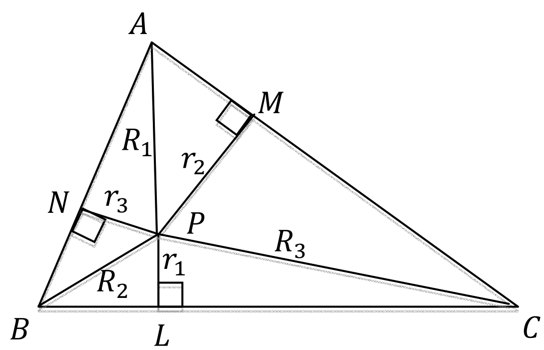

Figure 2.7: ΔABC with an interior point P

Now, consider quadrilateral $AMPN$ (see Figure 2.8 below).

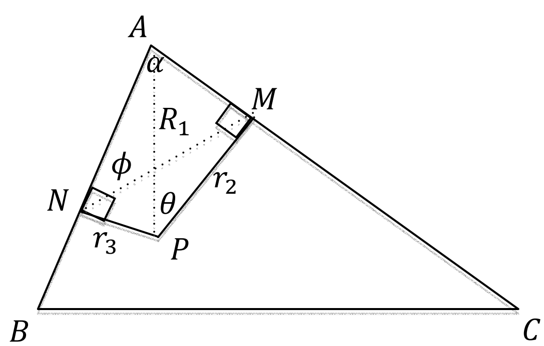

Figure 2.8: ΔABC with quadrilateral $AMPN$

We let $\angle BAC = \alpha$, $\angle APM = \theta$, and $\angle ANM = \phi$. Since $m\angle ANP = m\angle AMP = 90°$, then quadrilateral $AMPN$ is cyclic. Therefore, we can circumscribe a circle around points *A, M, P,* and *N*. In this circle, angles θ and ϕ cut off the same arc, so $\theta = \phi$. Applying the Law of Sines to ΔAMP, we see that

$$\frac{R_1}{\sin \angle AMP} = \frac{AM}{\sin \theta}.$$

But, $\sin \angle AMP = \sin 90° = 1$ and $\theta = \phi$, so we have

$$R_1 = \frac{AM}{\sin \phi}. \tag{2.3.1}$$

Applying the Law of Sines to ΔANM yields

$$\frac{NM}{\sin \alpha} = \frac{AM}{\sin \phi}. \tag{2.3.2}$$

And so, by (2.3.1) and (2.3.2), we get

$$R_1 = \frac{NM}{\sin \alpha},$$

which gives us

$$R_1 \sin \alpha = NM. \tag{2.3.3}$$

Since the angles in a quadrilateral must add to 360, $m\angle NPM + m\angle\alpha = 180°$. So, with the property that $\cos(180 - \omega) = -\cos(\omega)$ for any ω, $\cos \angle NPM = -\cos \alpha$. Then, applying the Law of Cosines to ΔNPM, we obtain,

$$\begin{aligned} NM^2 &= r_2^2 + r_3^2 - 2r_2r_3 \cos \angle MPN \\ &= r_2^2 + r_3^2 + 2r_2r_3 \cos \alpha. \end{aligned} \tag{2.3.4}$$

Suppose in ΔABC we call $\angle ABC$ and $\angle ACB$ β and γ, respectively. Then $\alpha + \beta + \gamma = 180$, so $\alpha = 180 - (\beta + \gamma)$. Again, with $\cos(180 - \omega) = -\cos(\omega)$, we then have

$$\begin{aligned} \cos\alpha = \cos[180 - (\beta+\gamma)] &= -\cos(\beta+\gamma) \\ &= -(\cos\beta\cos\gamma - \sin\beta\sin\gamma) \\ &= \sin\beta\sin\gamma - \cos\beta\cos\gamma. \qquad (2.3.5) \end{aligned}$$

So, substituting (2.3.5) in (2.3.4),

$$\begin{aligned} NM^2 &= r_2^2 + r_3^2 + 2r_2r_3(\sin\beta\sin\gamma - \cos\beta\cos\gamma) \\ &= r_2^2 + r_3^2 + 2r_2r_3\sin\beta\sin\gamma - 2r_2r_3\cos\beta\cos\gamma \\ &= r_2^2(\sin^2\gamma + \cos^2\gamma) + r_3^2(\sin^2\beta + \cos^2\beta) \\ &\quad + 2r_2r_3\sin\beta\sin\gamma - 2r_2r_3\cos\beta\cos\gamma \\ &= r_2^2\sin^2\gamma + 2r_2r_3\sin\beta\sin\gamma + r_3^2\sin^2\beta \\ &\quad + r_2^2\cos^2\gamma - 2r_2r_3\cos\beta\cos\gamma + r_3^2\cos^2\beta \\ &= (r_2\sin\gamma + r_3\sin\beta)^2 + (r_2\cos\gamma - r_3\cos\beta)^2 \\ &\geq (r_2\sin\gamma + r_3\sin\beta)^2. \end{aligned}$$

Hence,

$$NM \geq r_2\sin\gamma + r_3\sin\beta.$$

(Note here that we can square root both sides since we know $NM > 0$ and $\alpha, \beta, \gamma < 180°$, so $\sin\alpha, \sin\beta, \sin\gamma \geq 0$.) But, by (2.3.3) we have $NM = R_1\sin\alpha$. So,

$$R_1\sin\alpha \geq r_2\sin\gamma + r_3\sin\beta.$$

Therefore,

$$R_1 \geq r_2\left(\frac{\sin\gamma}{\sin\alpha}\right) + r_3\left(\frac{\sin\beta}{\sin\alpha}\right).$$

Following the analogous process for R_2 and R_3, we also have

$$R_2 \geq r_1\left(\frac{\sin\gamma}{\sin\beta}\right) + r_3\left(\frac{\sin\alpha}{\sin\beta}\right)$$

and

$$R_3 \geq r_1\left(\frac{\sin\beta}{\sin\gamma}\right) + r_2\left(\frac{\sin\alpha}{\sin\gamma}\right).$$

So,

$$R_1 + R_2 + R_3 \geq r_1\left(\frac{\sin\gamma}{\sin\beta} + \frac{\sin\beta}{\sin\gamma}\right) + r_2\left(\frac{\sin\gamma}{\sin\alpha} + \frac{\sin\alpha}{\sin\gamma}\right) + r_3\left(\frac{\sin\beta}{\sin\alpha} + \frac{\sin\alpha}{\sin\beta}\right).$$

For any positive number x, we know that $x + (1/x) \geq 2$. Applying this to the inequality above, we obtain our desired result:

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Finally, suppose ΔABC is equilateral with a side length of $2x$. Then, its height is $x\sqrt{3}$. Also, suppose P is the centroid. It is known that in an equilateral triangle, the distance from the centroid to the base of the triangle is 1/3 of the height. Hence, $r_1 = r_2 = r_3 = \left(x\sqrt{3}\right)/3$. Through simple trigonometry we also have $R_1 = R_2 = R_3 = \left(2x\sqrt{3}\right)/3$, and so we have equality above. ■

2.4 An inequality for the sides and inradius of a triangle

In 2012, Y. Wu and Srivastava (who is known to have published over 1,000 papers) proved the following conjecture of S. Wu [WS12].

Theorem 2.4.1. *Suppose* ΔABC *has sides of length a, b, and c and inradius r. Suppose also, without loss of generality, that* $0 \leq a \leq b \leq c$. *Then,*

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2} \leq \left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)-3\sqrt{3}\left(2-\sqrt{2}\right)r.$$

It turns out the proof of this theorem is quite tedious. We start by establishing the following three Lemmas.

Lemma 2.4.2. *Let* ΔABC *have sides of length a, b, and c and semi-perimeter s. Suppose, without loss of generality, that* $0 \leq a \leq b \leq c$. *Then,*

$$\sqrt{b^2+c^2}-\frac{\sqrt{2}}{2}(b+c) \leq \frac{\sqrt{6}}{4} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}.$$

Proof. From the Arithmetic-Geometric Mean Inequality (Theorem 2.1.1), we have

$$\sqrt{bc} \leq \frac{b+c}{2},$$

which implies

$$4bc \leq (b+c)^2 = b^2+2bc+c^2.$$

Therefore,

$$bc \leq \frac{b^2+c^2}{2}.$$

Adding $b^2/2$ and $c^2/2$ to both sides and simplifying, we get,

$$\frac{1}{2}(b+c)^2 \leq b^2+c^2.$$

So,

$$\frac{\sqrt{2}}{2}(b+c) \leq \sqrt{b^2+c^2}. \tag{2.4.1}$$

Next, notice that

$$\sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c)$$

$$= \left[\sqrt{b^2+c^2} - \left(\frac{\sqrt{2}}{2}\right)(b+c)\right] \frac{\left[\sqrt{b^2+c^2} + \left(\sqrt{2}/2\right)(b+c)\right]}{\left[\sqrt{b^2+c^2} + \left(\sqrt{2}/2\right)(b+c)\right]}$$

$$= \frac{(b^2+c^2) - (1/2)(b+c)^2}{\sqrt{b^2+c^2} + \left(\sqrt{2}/2\right)(b+c)}$$

$$= \frac{2(b^2+c^2) - (b+c)^2}{2\sqrt{b^2+c^2} + \sqrt{2}(b+c)}$$

$$= \frac{(b-c)^2}{2\sqrt{b^2+c^2} + \sqrt{2}(b+c)}$$

$$\leq \frac{(b-c)^2}{\sqrt{2}(b+c) + \sqrt{2}(b+c)}\text{ , by (2.4.1)}$$

$$= \frac{(b-c)^2}{2\sqrt{2}(b+c)}.$$

Hence,

$$\sqrt{b^2+c^2} - \left(\frac{\sqrt{2}}{2}\right)(b+c) \leq \frac{(b-c)^2}{2\sqrt{2}(b+c)}. \tag{2.4.2}$$

Again applying the Arithmetic-Geometric Mean Inequality (Theorem 2.1.1), we can see that

$$a = (s-b) + (s-c) \geq 2\sqrt{(s-b)(s-c)}. \tag{2.4.3}$$

So,

$$\sqrt{3}\sqrt{\frac{s-a}{s}} \cdot \frac{1}{a+a} \leq \sqrt{3}\sqrt{\frac{s-a}{s}} \cdot \frac{1}{a+2\sqrt{(s-b)(s-c)}},$$

or, equivalently,

$$\frac{\sqrt{3}}{2a}\sqrt{\frac{s-a}{s}} \leq \sqrt{3}\sqrt{\frac{s-a}{s}} \cdot \frac{1}{a+2\sqrt{(s-b)(s-c)}}. \tag{2.4.4}$$

Also, notice that

$$\frac{3(s-a)}{4a^2 s} - \frac{1}{(b+c)^2} = \frac{3(b+c-a)}{4a^2(a+b+c)} - \frac{1}{(b+c)^2}$$

$$= \frac{3(b+c-a)(b+c)^2 - 4a^2 s}{4a^2(b+c)^2(a+b+c)}$$

$$= \frac{(b+c-2a)[2a^2+3a(b+c)+3(b+c)^2]}{4a^2(b+c)^2(a+b+c)},$$

with some tedious computation. We assumed that $a \leq b \leq c$. So, $b+c-2a \geq 0$.

Therefore,

$$\frac{(b+c-2a)[2a^2+3a(b+c)+3(b+c)^2]}{4a^2(b+c)^2(a+b+c)} \geq 0,$$

and so,

$$\frac{1}{(b+c)^2} \leq \frac{3(s-a)}{4a^2 s}.$$

And then,

$$\frac{1}{b+c} \leq \frac{\sqrt{3}}{2a}\sqrt{\frac{s-a}{s}}. \tag{2.4.5}$$

Upon combining (2.4.4) and (2.4.5), we obtain

$$\frac{1}{b+c} \leq \sqrt{3}\sqrt{\frac{s-a}{s}} \cdot \frac{1}{a+2\sqrt{(s-b)(s-c)}}.$$

Notice that if we divide both sides of the above inequality by $2\sqrt{2}$ and multiply both sides by $(b-c)^2$, then

$$\frac{(b-c)^2}{2\sqrt{2}(b+c)} \leq \frac{\sqrt{6}}{4} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{\overline{a+2\sqrt{(s-b)(s-c)}}}. \tag{2.4.6}$$

Thus, by (2.4.2) and (2.4.6),

$$\sqrt{b^2+c^2} - \frac{\sqrt{2}}{2}(b+c) \leq \frac{\sqrt{6}}{4} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{\overline{a+2\sqrt{(s-b)(s-c)}}}. \quad \blacksquare$$

Lemma 2.4.3. *Again, let* ΔABC *have sides of length a, b, and c and semi-perimeter s. Suppose also, without loss of generality, that* $0 \leq a \leq b \leq c$. *Then,*

$$\sqrt{c^2+a^2} + \sqrt{a^2+b^2} - 2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} \leq \frac{\sqrt{6}}{8} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{\overline{a+2\sqrt{(s-b)(s-c)}}}.$$

Proof. Let $y = \sqrt{c^2+a^2}$ and $z = \sqrt{a^2+b^2}$. Since $a \leq b \leq c$, we then have

$$y = \sqrt{c^2+a^2} \leq \sqrt{c^2+c^2} = c\sqrt{2}$$

and

$$z = \sqrt{a^2+b^2} \leq \sqrt{b^2+b^2} = b\sqrt{2}\,.$$

So,

$$y \leq c\sqrt{2}$$

and

$$z \leq b\sqrt{2}\,.$$

Therefore,

$$(y+z)^2 = y^2 + 2yz + z^2$$

$$\leq 2c^2 + 4bc + 2b^2$$

$$= 2(b+c)^2. \tag{2.4.7}$$

Now,

$$(y+z)^2 = y^2 + 2yz + z^2$$

$$= 2y^2 + 2z^2 - (y^2 - 2yz + z^2)$$

$$= 2(y^2 + z^2) - (y-z)^2$$

$$= 2(c^2 + a^2 + a^2 + b^2) - \frac{(y-z)^2(y+z)^2}{(y+z)^2}$$

$$= 2(2a^2 + b^2 + c^2) - \frac{(y^2 - z^2)^2}{(y+z)^2}$$

$$= 2(2a^2 + b^2 + c^2) - \frac{(c^2 + a^2 - a^2 - b^2)^2}{(y+z)^2}$$

$$= 2(2a^2 + b^2 + c^2) - \frac{(b+c)^2(b-c)^2}{(y+z)^2}. \tag{2.4.8}$$

Then (2.4.7) together with (2.4.8) gives us

$$2(2a^2 + b^2 + c^2) - \frac{(b+c)^2(b-c)^2}{(y+z)^2}$$

$$\leq 2(2a^2 + b^2 + c^2) - \frac{(b+c)^2(b-c)^2}{2(b+c)^2}$$

$$= 4a^2 + 2b^2 + 2c^2 - \frac{(b-c)^2}{2}$$

$$= 4a^2 + 2b^2 + 2c^2 - \frac{b^2}{2} + bc - \frac{c^2}{2}$$

$$= 4a^2 + b^2 + 2bc + c^2 + \frac{b^2}{2} - bc + \frac{c^2}{2}$$

$$= 4a^2 + (b+c)^2 + \frac{(b-c)^2}{2}. \tag{2.4.9}$$

Hence, by (2.4.8) and (2.4.9),

$$(y+z)^2 \le 4a^2 + (b+c)^2 + \frac{(b-c)^2}{2}.$$

Next we square root and use that $y = \sqrt{c^2 + a^2}, z = \sqrt{a^2 + b^2}$. This yields

$$\sqrt{c^2 + a^2} + \sqrt{a^2 + b^2} \le \sqrt{4a^2 + (b+c)^2 + \frac{(b-c)^2}{2}}.$$

Subtracting equal values from both sides we get

$$\sqrt{c^2 + a^2} + \sqrt{a^2 + b^2} - 2\sqrt{a^2 + \left(\frac{b+c}{2}\right)^2}$$

$$\le \sqrt{4a^2 + (b+c)^2 + \frac{(b-c)^2}{2}} - \sqrt{4a^2 + (b+c)^2}.$$

Now we multiply by its conjugate over its conjugate and have

$$\sqrt{c^2 + a^2} + \sqrt{a^2 + b^2} - 2\sqrt{a^2 + \left(\frac{b+c}{2}\right)^2}$$

$$\le \frac{4a^2 + (b+c)^2 + [(b-c)^2/2] - 4a^2 - (b+c)^2}{\sqrt{4a^2 + (b+c)^2 + [(b-c)^2/2]} + \sqrt{4a^2 + (b+c)^2}}$$

$$= \frac{[(b-c)^2/2]}{\sqrt{4a^2 + (b+c)^2 + [(b-c)^2/2]} + \sqrt{4a^2 + (b+c)^2}}$$

$$\le \frac{[(b-c)^2/2]}{\sqrt{4a^2 + (b+c)^2} + \sqrt{4a^2 + (b+c)^2}}$$

$$= \frac{(b-c)^2}{4\sqrt{4a^2+(b+c)^2}}. \tag{2.4.10}$$

Next, notice

$$\frac{3(s-a)}{8a^2s} - \frac{1}{4a^2+(b+c)^2}$$

$$= \frac{3(b+c-a)}{8a^2(a+b+c)} - \frac{1}{4a^2+(b+c)^2}$$

$$= \frac{3(b+c-a)[4a^2+(b+c)^2] - 8a^2(a+b+c)}{8a^2(a+b+c)[4a^2+(b+c)^2]}$$

$$= \frac{(b+c-2a)[10a^2+3(b+c)a+3(b+c)^2]}{8a^2(a+b+c)[4a^2+(b+c)^2]},$$

with some tedious computation. Recall we assumed that $a \leq b \leq c$ and so $b+c-2a \geq 0$. Therefore,

$$\frac{(b+c-2a)[10a^2+3(b+c)a+3(b+c)^2]}{8a^2(a+b+c)[4a^2+(b+c)^2]} \geq 0.$$

Hence,

$$\frac{1}{4a^2+(b+c)^2} \leq \frac{3(s-a)}{8a^2s},$$

which is equivalent to

$$\frac{1}{\sqrt{4a^2+(b+c)^2}} \leq \frac{\sqrt{6}}{4} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{1}{a},$$

and also equivalent to

$$\frac{(b-c)^2}{4\sqrt{4a^2+(b+c)^2}} \leq \frac{\sqrt{6}}{16} \cdot \sqrt{\frac{s-a}{s}} \cdot \frac{(b-c)^2}{a}. \tag{2.4.11}$$

Finally, by (2.4.3),

$$\frac{\sqrt{6}}{8}\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+a} \leq \frac{\sqrt{6}}{8}\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}},$$

or,

$$\frac{\sqrt{6}}{16}\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a} \leq \frac{\sqrt{6}}{8}\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}. \qquad (2.4.12)$$

Therefore, from (2.4.10) – (2.4.12) we have

$$\sqrt{c^2+a^2}+\sqrt{a^2+b^2}-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2} \leq \frac{\sqrt{6}}{8}\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}},$$

our desired result. ■

Lemma 2.4.4. *Again, let* ΔABC *have sides of length a, b, and c and semi-perimeter s. Suppose also, without loss of generality, that* $0 \leq a \leq b \leq c$. *Let*

$$f(a,b,c) \coloneqq \sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}$$
$$-\left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)+3\sqrt{3}\left(2-\sqrt{2}\right)\cdot\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

Then,

$$f(a,b,c) \leq f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right).$$

Proof. Notice

$$f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right) \coloneqq \sqrt{a^2+\left(\frac{b+c}{2}\right)^2}+\sqrt{\left(\frac{b+c}{2}\right)^2+\left(\frac{b+c}{2}\right)^2}+\sqrt{\left(\frac{b+c}{2}\right)^2+a^2}$$
$$-\left(1+\frac{\sqrt{2}}{2}\right)\left(a+\frac{b+c}{2}+\frac{b+c}{2}\right)$$

$$+3\sqrt{3}(2-\sqrt{2})\cdot\frac{\sqrt{s(s-a)[s-(b+c)/2][s-(b+c)/2]}}{s}$$

$$=2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}+\frac{\sqrt{2}}{2}(b+c)$$

$$-\left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)$$

$$+\left[\frac{3\sqrt{3}(2-\sqrt{2})}{2}\right]\cdot a\sqrt{\frac{(s-a)}{s}}\,. \tag{2.4.13}$$

So, by (2.4.13) we are equivalently trying to prove

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}-\left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)$$

$$+3\sqrt{3}(2-\sqrt{2})\cdot\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

$$\leq 2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}+\frac{\sqrt{2}}{2}(b+c)-\left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)$$

$$+\left[\frac{3\sqrt{3}(2-\sqrt{2})}{2}\right]\cdot a\sqrt{\frac{(s-a)}{s}},$$

or,

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}-\frac{\sqrt{2}}{2}(b+c)-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}$$

$$\leq\left[\frac{3\sqrt{3}(2-\sqrt{2})}{2}\right]\sqrt{\frac{(s-a)}{s}}\,.\left[a-2\sqrt{(s-b)(s-c)}\right]. \tag{2.4.14}$$

Now notice that, with some algebra,

$$\left[a-2\sqrt{(s-b)(s-c)}\right]\left[\frac{a+2\sqrt{(s-b)(s-c)}}{a+2\sqrt{(s-b)(s-c)}}\right]=\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}.$$

This, together with (2.4.14), and we are equivalently trying to prove

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}-\frac{\sqrt{2}}{2}(b+c)-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}$$

$$\leq\left[\frac{3\sqrt{3}\left(2-\sqrt{2}\right)}{2}\right]\sqrt{\frac{(s-a)}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}. \qquad (2.4.15)$$

Hence, if we prove (2.4.15), we have achieved our desired result.

By Lemmas 2.4.2 and 2.4.3,

$$\sqrt{b^2+c^2}-\frac{\sqrt{2}}{2}(b+c)+\sqrt{c^2+a^2}+\sqrt{a^2+b^2}-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}$$

$$\leq\left(\frac{3\sqrt{6}}{8}\right)\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}}$$

Since $\left(3\sqrt{6}/8\right)<\left[3\sqrt{3}(2-\sqrt{2})/2\right]$, then

$$\sqrt{b^2+c^2}-\frac{\sqrt{2}}{2}(b+c)+\sqrt{c^2+a^2}+\sqrt{a^2+b^2}-2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}$$

$$\leq\left[\frac{3\sqrt{3}\left(2-\sqrt{2}\right)}{2}\right]\sqrt{\frac{s-a}{s}}\cdot\frac{(b-c)^2}{a+2\sqrt{(s-b)(s-c)}},$$

which is precisely 2.4.15). ■

With the above lemmas established, we are now ready to prove Theorem 2.4.1. Note first that by Lemma 2.2.2, $r=(\Delta/s)$ and by Heron's formula (Theorem

2.2.1), $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where Δ denotes the area of ΔABC. So, for Theorem 2.4.1, we are proving

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}-\left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)$$

$$+3\sqrt{3}\left(2-\sqrt{2}\right)\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \leq 0. \tag{2.4.16}$$

The left-hand side of (2.4.16) is precisely $f(a,b,c)$, and from Lemma 2.4.4, we have that $f(a,b,c) \leq f(a,(b+c)/2\,,(b+c)/2)$. So, if we can show that $f(a,(b+c)/2\,,(b+c)/2) \leq 0$, then we will have what we need. Therefore, by (2.4.13), we are now working to prove that

$$2\sqrt{a^2+\left(\frac{b+c}{2}\right)^2}+\frac{\sqrt{2}}{2}(b+c)-\left(1+\frac{\sqrt{2}}{2}\right)(a+b+c)$$

$$+\left[\frac{3\sqrt{3}\left(2-\sqrt{2}\right)}{2}\right]a\sqrt{\frac{(s-a)}{s}} \leq 0. \tag{2.4.17}$$

Without loss of generality, we let $a = 1$ and $[(b+c)/2] = x$. (Note that $x \geq 1$ since $a \leq b \leq c$.) Then, (2.4.17) becomes

$$2\sqrt{1+x^2}+x\sqrt{2}-\left(1+\frac{\sqrt{2}}{2}\right)(1+2x)+\left[\frac{3\sqrt{3}\left(2-\sqrt{2}\right)}{2}\right]\sqrt{\frac{s-1}{s}}$$

$$= 2\sqrt{1+x^2}+x\sqrt{2}-1-2x-\frac{\sqrt{2}}{2}-x\sqrt{2}+\left[\frac{3\sqrt{3}\left(2-\sqrt{2}\right)}{2}\right]\sqrt{\frac{s-1}{s}}$$

$$= 2\sqrt{1+x^2}-2x-\left(1+\frac{\sqrt{2}}{2}\right)+\left[\frac{3\sqrt{3}\left(2-\sqrt{2}\right)}{2}\right]\sqrt{\frac{s-1}{s}} \leq 0. \tag{2.4.18}$$

Notice,

$$\frac{s-1}{s} = \frac{\frac{a+b+c}{2}-1}{\frac{a+b+c}{2}} = \frac{a+b+c-2}{a+b+c} = \frac{2x-1}{2x+1}.$$

Applying this to (2.4.18), we see that we are now equivalently proving

$$g(x) = 2\sqrt{1+x^2} - 2x - \left(1+\frac{\sqrt{2}}{2}\right) + \left[\frac{3\sqrt{3}(2-\sqrt{2})}{2}\right] \cdot \sqrt{\frac{2x-1}{2x+1}} \leq 0, \quad (2.4.19)$$

where $x \geq 1$. Since $\sqrt{(2x-1)/(2x+1)} < 1$, we have that

$$g(x) < \; 2\sqrt{1+x^2} - 2x - \left(1+\frac{\sqrt{2}}{2}\right) + \left[\frac{3\sqrt{3}(2-\sqrt{2})}{2}\right]. \quad (2.4.20)$$

Now,

$$\begin{aligned} 2\sqrt{1+x^2} - 2x &= 2\left(\sqrt{1+x^2} - x\right) \\ &= 2\left(\sqrt{1+x^2} - x\right)\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right) \\ &= 2\left(\frac{1+x^2-x^2}{\sqrt{1+x^2}+x}\right) \\ &= 2\left(\frac{1}{\sqrt{1+x^2}+x}\right) \\ &< 2\left(\frac{1}{\sqrt{x^2}+x}\right) \\ &= \frac{1}{x}, \end{aligned} \quad (2.4.21)$$

So, (2.4.20) together with (2.4.21) gives us

$$g(x) < \; \frac{1}{x} - \left(1+\frac{\sqrt{2}}{2}\right) + \frac{3\sqrt{3}(2-\sqrt{2})}{2} \approx \frac{1}{x} - 0.18.$$

Then, $g(x) \leq 0$ when $(1/x) - 0.18 \leq 0$, or when $x \geq 5.6$. We have left to show then that $g(x) \leq 0$ for $1 \leq x < 5.6$. Figure 2.9 below is the graph of $g(x)$ for $1 \leq x \leq 10$. Clearly from the graph, $g(x) \leq 0$ for these values of x as well. This completes our proof. ■

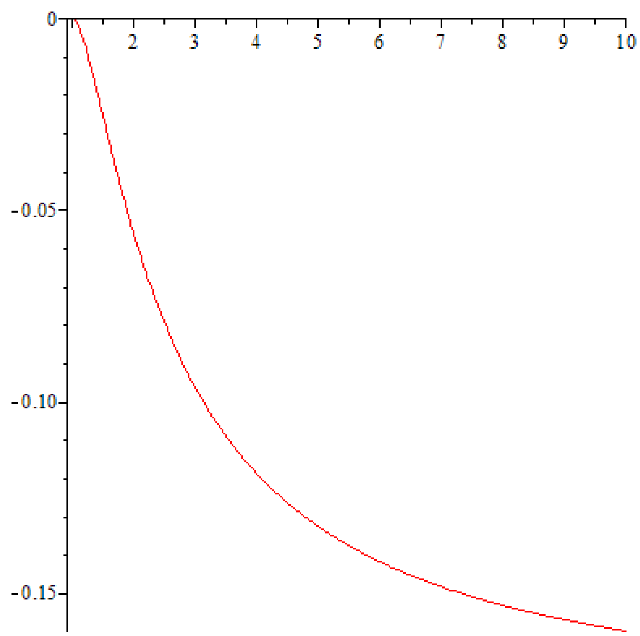

Figure 2.9: Graph of $g(x)$ for $1 \leq x \leq 10$.

2.5 Inequalities for areas associated with conics

It has long been known that the area of the triangle formed by three tangent lines to a parabola is equal to half that of the triangle formed by joining their points of tangency. In 1991, Day found equally beautiful results for both the ellipse and the hyperbola [Day91]. The area of the triangle formed by three tangent lines to an ellipse is strictly greater than half that of the triangle formed by joining their points of tangency. The area of the triangle formed by three tangent lines to a hyperbola is strictly less than half that of the triangle formed by joining their points of tangency (provided the three points lie on the same branch of the hyperbola.) We will explore his proofs for the parabola and ellipse below, omitting the proof for the hyperbola, because it is very similar to that for the ellipse. We begin with an important lemma.

Lemma 2.5.1. *Let* $(x_1, y_1), (x_2, y_2)$, *and* (x_3, y_3) *be any three points in the plane. The area of the triangle,* Δ, *formed by joining the three points is*

$$\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|.$$

Proof. Let $P = (x_1, y_1), Q = (x_2, y_2)$, and $R = (x_3, y_3)$ be any three points in the plane. Also let

$$\overrightarrow{PQ} = \boldsymbol{u} = (x_2 - x_1)\boldsymbol{i} + (y_2 - y_1)\boldsymbol{j} + 0\boldsymbol{k},$$

and

$$\overrightarrow{PR} = \boldsymbol{v} = (x_3 - x_1)\boldsymbol{i} + (y_3 - y_1)\boldsymbol{j} + 0\boldsymbol{k}.$$

Δ will denote the area of ΔPQR. It is known that $\Delta = (1/2)\|\boldsymbol{u} \times \boldsymbol{v}\|$, where $\|\boldsymbol{u}\|$ is the Euclidian norm of vector $\boldsymbol{u}$. So,

$$\Delta = \frac{1}{2}\left\|\begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{matrix}\right\|$$

$$= \frac{1}{2}\left| |\boldsymbol{i}| \begin{vmatrix} y_2 - y_1 & 0 \\ y_3 - y_1 & 0 \end{vmatrix} - |\boldsymbol{j}| \begin{vmatrix} x_2 - x_1 & 0 \\ x_3 - x_1 & 0 \end{vmatrix} + |\boldsymbol{k}| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \right|$$

$$= \left| \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \right|$$

$$= \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \right|.$$

We can add and subtract rows in any matrix without changing the value of the determinant. So, we add row 1 to row 2, replacing row 2 by the result. Similarly, we add row 1 to row 3, replacing row 3 by the result. Hence,

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|,$$

Finally, switching rows and columns does not change the value of the determinant either. Therefore,

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \right| = \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|,$$

as desired. ■

Theorem 2.5.2. *The area of the triangle formed by three tangent lines to a parabola is equal to half that of the triangle formed by joining their points of tangency.*

Proof. Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be three points on the parabola with equation $y^2 = 4ax$, where $a > 0$. Let the area of the triangle formed by joining these three points be equal to Δ. By Lemma 2.5.1,

$$\Delta = \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$= \frac{1}{2}\left|\frac{y_1^2}{4a}(y_2 - y_3) + \frac{y_2^2}{4a}(y_3 - y_1) + \frac{y_3^2}{4a}(y_1 - y_2)\right|.$$

Straightforward manipulation leads to

$$\Delta = \frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|. \tag{2.5.1}$$

Next we will find the equations of the tangent lines to the parabola at the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Again, for the parabola, $y^2 = 4ax$. Then, $y'(x_1) = 2a/y_1$, and so we have

$$y - y_1 = \frac{2a}{y_1}(x - x_1),$$

or

$$yy_1 - y_1^2 = 2ax - 2ax_1.$$

Then,

$$yy_1 - 4ax_1 = 2ax - 2ax_1,$$

and so

$$y = \frac{2a(x + x_1)}{y_1}.$$

The same process is followed to get the equations of the tangent lines at (x_2, y_2) and (x_3, y_3). Hence, the equations of our three tangent lines are the following:

$$y = \frac{2a(x + x_1)}{y_1}, \quad y = \frac{2a(x + x_2)}{y_2}\ , \quad y = \frac{2a(x + x_3)}{y_3}\ .$$

And with some simple algebra, we find their three points of intersection to be

$$\left(\frac{y_1 y_2}{4a}, \frac{y_1 + y_2}{2}\right), \quad \left(\frac{y_2 y_3}{4a}, \frac{y_2 + y_3}{2}\right), \quad \left(\frac{y_3 y_1}{4a}, \frac{y_3 + y_1}{2}\right).$$

Let Δ' be the area of the triangle formed by the points of intersection of the tangent lines. Then by Lemma 2.5.2,

$$\Delta' = \frac{1}{2}\left|\frac{y_1y_2}{4a}\left(\frac{y_2+y_3}{2}-\frac{y_3+y_1}{2}\right)+\frac{y_2y_3}{4a}\left(\frac{y_3+y_1}{2}-\frac{y_1+y_2}{2}\right)\right.$$
$$\left.+\frac{y_3y_1}{4a}\left(\frac{y_1+y_2}{2}-\frac{y_2+y_3}{2}\right)\right|$$
$$=\frac{1}{16a}|y_1y_2(y_2-y_1)+y_2y_3(y_3-y_2)+y_3y_1(y_1-y_3)|$$
$$=\frac{1}{16a}|(y_1-y_2)(y_2-y_3)(y_3-y_1)|$$
$$=\frac{1}{2}\Delta\,,$$

by (2.5.1). Therefore, the area of the triangle formed by three tangent lines to a parabola is equal to half that of the triangle formed by joining their points of tangency. ■

Before we begin our proof for the ellipse, we prove the following lemma.

Lemma 2.5.3. *Let α, β, and γ be any real numbers. Then*

$$(\cos^2\alpha\cos^2\beta\cos^2\gamma)^{\frac{1}{3}}+(\sin^2\alpha\sin^2\beta\sin^2\gamma)^{\frac{1}{3}}\le 1.$$

Proof. Suppose *a, b, c, d, e, f* are any nonnegative real numbers. Observe that upon multiplication,

$$\left[(abc)^{\frac{1}{3}}+(def)^{\frac{1}{3}}\right]^3=(abc)+3(a^2b^2c^2def)^{\frac{1}{3}}+3(abcd^2e^2f^2)^{\frac{1}{3}}+(def).$$

But, by the Arithmetic-Geometric Mean Inequality (Theorem 2.1.1),

$$3(a^2b^2c^2def)^{\frac{1}{3}}\le abf+bcd+ace$$

and

$$3(abcd^2e^2f^2)^{\frac{1}{3}} \leq aef + bdf + cde.$$

Hence,

$$\left[(abc)^{\frac{1}{3}} + (def)^{\frac{1}{3}}\right]^3 \leq (abc) + (abf + bcd + ace\ ) + (aef + bdf + cde) + (def)$$

$$= (a + d)(b + e)(c + f).$$

Therefore,

$$(abc)^{\frac{1}{3}} + (def)^{\frac{1}{3}} \leq (a + d)^{\frac{1}{3}}(b + e)^{\frac{1}{3}}(c + f)^{\frac{1}{3}}. \qquad (2.5.2)$$

Now, let α, β, and γ be real numbers so that

$$a = \cos^2\alpha, \quad b = \cos^2\beta, \quad c = \cos^2\gamma,$$

$$d = \sin^2\alpha, \quad e = \sin^2\beta, \quad f = \sin^2\gamma.$$

Then, by (2.5.2),

$$(\cos^2\alpha\cos^2\beta\cos^2\gamma)^{\frac{1}{3}} + (\sin^2\alpha\sin^2\beta\sin^2\gamma)^{\frac{1}{3}}$$

$$\leq (\cos^2\alpha + \sin^2\alpha)^{\frac{1}{3}}(\cos^2\beta + \sin^2\beta)^{\frac{1}{3}}(\cos^2\gamma + \sin^2\gamma)^{\frac{1}{3}}$$

$$= 1. \quad \blacksquare$$

Note the following well-known trigonometric identities, which will be used in our proof of Theorem 2.5.4.

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right). \qquad (2.5.3)$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta. \qquad (2.5.4)$$

$$\sin\theta \pm \sin\phi = 2\sin\left(\frac{\theta \pm \phi}{2}\right)\cos\left(\frac{\theta \mp \phi}{2}\right). \tag{2.5.5}$$

$$\cos\theta \pm \cos\phi = 2\sin\left(\frac{\theta \pm \phi}{2}\right)\sin\left(\frac{\phi \mp \theta}{2}\right). \tag{2.5.6}$$

Theorem 2.5.4. *The area of the triangle formed by three tangent lines to an ellipse is strictly greater than half that of the triangle formed by joining their points of tangency.*

Proof. For this proof, we will use the parametric equations for the ellipse. We let $(a\cos t_1, b\sin t_1)$, $(a\cos t_2, b\sin t_2)$, and $(a\cos t_3, b\sin t_3)$ be three points on the ellipse, with $a, b > 0$. We also let Δ denote the area of the triangle formed by joining these three points. Then, by Lemma 2.5.1,

$$\Delta = \frac{1}{2}ab|\cos t_1(\sin t_2 - \sin t_3) + \cos t_2(\sin t_3 - \sin t_1)$$

$$+ \cos t_3(\sin t_1 - \sin t_2)|$$

$$= \frac{1}{2}ab|(\cos t_1 \sin t_2 - \cos t_2 \sin t_1) + (\cos t_2 \sin t_3 - \cos t_3 \sin t_2)$$

$$+ \cos t_3 \sin t_1 - \cos t_1 \sin t_3|$$

$$= \frac{1}{2}ab|\sin(t_2 - t_1) + \sin(t_3 - t_2) + \sin(t_1 - t_3)|, \text{by (2.5.4)}$$

$$= \frac{1}{2}ab|\sin(t_1 - t_2) + \sin(t_2 - t_3) + \sin(t_3 - t_1)|$$

$$= \frac{1}{2}ab\left|2\sin\left(\frac{t_1 - t_2}{2}\right)\cos\left(\frac{t_1 - t_2}{2}\right) + \sin(t_2 - t_3) + \sin(t_3 - t_1)\right|, \text{by (2.5.3)}$$

$$= \frac{1}{2}ab\left|2\sin\left(\frac{t_1 - t_2}{2}\right)\cos\left(\frac{t_1 - t_2}{2}\right) + 2\sin\left(\frac{t_2 - t_1}{2}\right)\cos\left(\frac{t_1 + t_2 - 2t_3}{2}\right)\right|, \text{by (2.5.5)}$$

$$= ab\left|\sin\left(\frac{t_1 - t_2}{2}\right)\cos\left(\frac{t_1 - t_2}{2}\right) - \sin\left(\frac{t_1 - t_2}{2}\right)\cos\left(\frac{t_1 + t_2 - 2t_3}{2}\right)\right|$$

$$= ab\left|\sin\left(\frac{t_1 - t_2}{2}\right)\left[\cos\left(\frac{t_1 - t_2}{2}\right) - \cos\left(\frac{t_1 + t_2 - 2t_3}{2}\right)\right]\right|$$

$$= ab\left|\sin\left(\frac{t_1 - t_2}{2}\right)\left[-2\sin\left(\frac{t_1 - t_3}{2}\right)\sin\left(\frac{t_3 - t_2}{2}\right)\right]\right|, \text{by (2.5.6)}$$

$$= 2ab\left|\sin\left(\frac{t_1 - t_2}{2}\right)\sin\left(\frac{t_2 - t_3}{2}\right)\sin\left(\frac{t_3 - t_1}{2}\right)\right|. \qquad (2.5.7)$$

Next we will find the equations of the tangent lines to the ellipse at the points $(a\cos t_1, b\sin t_1)$, $(a\cos t_2, b\sin t_2)$, and $(a\cos t_3, b\sin t_3)$. First consider the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiating, we have

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y},$$

and then the equation of the tangent line at (x_1, y_1) takes the form

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

Therefore,

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{x_1^2}{a^2} - \frac{xx_1}{a^2},$$

and so,

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

With $(x_1, y_1) = (a\cos t_1, b\sin t_1)$, the above equation becomes

$$\frac{x\cos t_1}{a}+\frac{y\sin t_1}{b}=1.$$

The same process is followed to get the equations of the lines tangent to the ellipse at the other two points. Hence, the equations of our tangent lines are the following:

$$\frac{x\cos t_1}{a}+\frac{y\sin t_1}{b}=1,\quad \frac{x\cos t_2}{a}+\frac{y\sin t_2}{b}=1,\quad \frac{x\cos t_3}{a}+\frac{y\sin t_3}{b}=1.$$

To simplify our algebra for finding the points of intersection of the lines above, let $\cos t_i = c_i$ and $\sin t_i = s_i$. We also let $\cos(t_i - t_j) = c_{ij}$ and $\sin(t_i - t_j) = s_{ij}$. Upon applying this notation, we have the following for the equations of our three tangent lines:

$$\frac{x\,c_1}{a}+\frac{y\,s_1}{b}=1. \tag{2.5.8}$$

$$\frac{x\,c_2}{a}+\frac{y\,s_2}{b}=1. \tag{2.5.9}$$

$$\frac{x\,c_3}{a}+\frac{y\,s_3}{b}=1. \tag{2.5.10}$$

We can now much more clearly find the points of intersection. We start by subtracting c_1 times equation (2.5.9) from c_2 times equation (2.5.8). We obtain

$$\frac{y\,c_2 s_1}{b}-\frac{y\,c_1 s_2}{b}=c_2-c_1.$$

So,

$$\begin{aligned} y &= \frac{-b(c_1-c_2)}{c_2 s_1 - c_1 s_2} \\ &= \frac{-b(c_1-c_2)}{s_{12}}, \text{ by (2.5.4).} \end{aligned}$$

Next, we subtract s_1 times equation (2.5.9) from s_2 times equation (2.5.8). This gives us

$$\frac{x\,c_1 s_2}{a} - \frac{x\,c_2 s_1}{a} = s_2 - s_1.$$

So,

$$x = \frac{a(s_1 - s_2)}{c_2 s_1 - c_1 s_2}$$

$$= \frac{a(s_1 - s_2)}{s_{12}}, \text{by (2.5.4)}.$$

The same process is applied to get the other two points. Hence, our three points of intersection are the following:

$$\left(\frac{a(s_1 - s_2)}{s_{12}}, \frac{-b(c_1 - c_2)}{s_{12}}\right), \left(\frac{a(s_2 - s_3)}{s_{23}}, \frac{-b(c_2 - c_3)}{s_{23}}\right), \left(\frac{a(s_3 - s_1)}{s_{31}}, \frac{-b(c_3 - c_1)}{s_{31}}\right).$$

Let Δ' denote the area of the triangle formed by the three points of intersection. Then, by Lemma 2.5.1,

$$\begin{aligned}
\Delta' = \frac{1}{2}\Bigg| &\frac{a(s_1 - s_2)}{s_{12}}\left(\frac{-b(c_2 - c_3)}{s_{23}} + \frac{b(c_3 - c_1)}{s_{31}}\right) \\
&+ \frac{a(s_2 - s_3)}{s_{23}}\left(\frac{-b(c_3 - c_1)}{s_{31}} + \frac{b(c_1 - c_2)}{s_{12}}\right) \\
&+ \frac{a(s_3 - s_1)}{s_{31}}\left(\frac{-b(c_1 - c_2)}{s_{12}} + \frac{b(c_2 - c_3)}{s_{23}}\right)\Bigg| \\
= \frac{ab}{2}\Bigg| &\frac{-s_{31}(s_1 - s_2)(c_2 - c_3) + s_{23}(s_1 - s_2)(c_3 - c_1)}{s_{12}s_{23}s_{31}} \\
&+ \frac{-s_{12}(s_2 - s_3)(c_3 - c_1) + s_{31}(s_2 - s_3)(c_1 - c_2)}{s_{12}s_{23}s_{31}}
\end{aligned}$$

$$+\frac{-s_{23}(s_3-s_1)(c_1-c_2)+s_{12}(s_3-s_1)(c_2-c_3)}{s_{12}s_{23}s_{31}}\Bigg|$$

$$=\frac{ab}{2}\Bigg|\frac{s_{12}[(s_3-s_1)(c_2-c_3)-(s_2-s_3)(c_3-c_1)]}{s_{12}s_{23}s_{31}}$$

$$+\frac{s_{23}[(s_1-s_2)(c_3-c_1)-(s_3-s_1)(c_1-c_2)]}{s_{12}s_{23}s_{31}}$$

$$+\frac{s_{31}[(s_2-s_3)(c_1-c_2)-(s_1-s_2)(c_2-c_3)]}{s_{12}s_{23}s_{31}}\Bigg|$$

$$=\frac{ab}{2}\Bigg|\frac{s_{12}[(c_2s_3-c_3s_2)+(c_3s_1-c_1s_3)+(c_1s_2-c_2s_1)]}{s_{12}s_{23}s_{31}}$$

$$+\frac{s_{23}[(c_2s_3-c_3s_2)+(c_3s_1-c_1s_3)+(c_1s_2-c_2s_1)]}{s_{12}s_{23}s_{31}}$$

$$+\frac{s_{31}[(c_2s_3-c_3s_2)+(c_3s_1-c_1s_3)+(c_1s_2-c_2s_1)]}{s_{12}s_{23}s_{31}}\Bigg|$$

$$=\frac{ab}{2}\Bigg|\frac{s_{12}(s_{21}+s_{32}+s_{13})}{s_{12}s_{23}s_{31}}+\frac{s_{23}(s_{21}+s_{32}+s_{13})}{s_{12}s_{23}s_{31}}$$

$$+\frac{s_{31}(s_{21}+s_{32}+s_{13})}{s_{12}s_{23}s_{31}}\Bigg|, \text{by (2.5.4)}$$

$$=\frac{ab}{2}\Bigg|\frac{(s_{12}+s_{23}+s_{31})(s_{21}+s_{32}+s_{13})}{s_{12}s_{23}s_{31}}\Bigg|$$

$$=\frac{ab}{2}\Bigg|\frac{(s_{12}+s_{23}+s_{31})(-s_{12}-s_{23}-s_{31})}{s_{12}s_{23}s_{31}}\Bigg|$$

$$=\frac{ab|s_{12}+s_{23}+s_{31}|^2}{2|s_{12}s_{23}s_{31}|}$$

$$=\frac{ab|\sin(t_1-t_2)+\sin(t_2-t_3)+\sin(t_3-t_1)|^2}{2|\sin(t_1-t_2)\sin(t_2-t_3)\sin(t_3-t_1)|}$$

$$= \frac{ab\left|2\sin\left(\frac{t_1-t_2}{2}\right)\cos\left(\frac{t_1-t_2}{2}\right)+\sin(t_2-t_3)+\sin(t_3-t_1)\right|^2}{2|\sin(t_1-t_2)\sin(t_2-t_3)\sin(t_3-t_1)|} \text{ , by (2.5.3)}$$

$$= \frac{ab\left|2\sin\left(\frac{t_1-t_2}{2}\right)\cos\left(\frac{t_1-t_2}{2}\right)+2\sin\left(\frac{t_1-t_2}{2}\right)\cos\left(\frac{t_1+t_2-2t_3}{2}\right)\right|^2}{2|\sin(t_1-t_2)\sin(t_2-t_3)\sin(t_3-t_1)|} \text{ , by (2.5.5)}$$

$$= \frac{2ab\left|\sin\left(\frac{t_1-t_2}{2}\right)\left[\cos\left(\frac{t_1-t_2}{2}\right)+\cos\left(\frac{t_1+t_2-2t_3}{2}\right)\right]\right|^2}{|\sin(t_1-t_2)\sin(t_2-t_3)\sin(t_3-t_1)|}$$

$$= \frac{2ab\left|\sin\left(\frac{t_1-t_2}{2}\right)\left[2\sin\left(\frac{t_1-t_3}{2}\right)\sin\left(\frac{t_2-t_3}{2}\right)\right]\right|^2}{|\sin(t_1-t_2)\sin(t_2-t_3)\sin(t_3-t_1)|} \text{ , by (2.5.6)}$$

$$= \frac{8ab\left|\sin\left(\frac{t_1-t_2}{2}\right)\sin\left(\frac{t_2-t_3}{2}\right)\sin\left(\frac{t_3-t_1}{2}\right)\right|^2}{|\sin(t_1-t_2)\sin(t_2-t_3)\sin(t_3-t_1)|}.$$

To simplify the next few steps, we now let $\alpha = t_1 - t_2$, $\beta = t_2 - t_3$, and $\gamma = t_3 - t_1$. Then,

$$\Delta' = \frac{8ab\left|\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\right|^2}{|\sin\alpha\sin\beta\sin\gamma|}$$

$$= 8ab\left|\left[\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\alpha}\right]\left[\sin\left(\frac{\alpha}{2}\right)\right]\left[\frac{\sin\left(\frac{\beta}{2}\right)}{\sin\beta}\right]\left[\sin\left(\frac{\beta}{2}\right)\right]\left[\frac{\sin\left(\frac{\gamma}{2}\right)}{\sin\gamma}\right]\left[\sin\left(\frac{\gamma}{2}\right)\right]\right|$$

$$= 8ab\left|\left[\frac{1}{2\cos\left(\frac{\alpha}{2}\right)}\right]\left[\sin\left(\frac{\alpha}{2}\right)\right]\left[\frac{1}{2\cos\left(\frac{\beta}{2}\right)}\right]\left[\sin\left(\frac{\beta}{2}\right)\right]\left[\frac{1}{2\cos\left(\frac{\gamma}{2}\right)}\right]\left[\sin\left(\frac{\gamma}{2}\right)\right]\right| \text{, by (2.5.3)}$$

$$= ab\left|\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right)\right|.$$

Therefore,

$$\Delta' = ab \left|\tan\left(\frac{t_1 - t_2}{2}\right)\tan\left(\frac{t_2 - t_3}{2}\right)\tan\left(\frac{t_3 - t_1}{2}\right)\right|. \tag{2.5.11}$$

Hence, by (2.5.7) and (2.5.11),

$$\frac{\Delta}{2\Delta'} = \left|\cos\left(\frac{t_1 - t_2}{2}\right)\cos\left(\frac{t_2 - t_3}{2}\right)\cos\left(\frac{t_3 - t_1}{2}\right)\right|. \tag{2.5.12}$$

Notice also that by (2.5.7) we have

$$\frac{\Delta}{2ab} = \left|\sin\left(\frac{t_1 - t_2}{2}\right)\sin\left(\frac{t_2 - t_3}{2}\right)\sin\left(\frac{t_3 - t_1}{2}\right)\right|. \tag{2.5.13}$$

So, by Lemma 2.5.3 together with (2.5.12) and (2.5.13), we have

$$\left(\frac{\Delta}{2\Delta'}\right)^{2/3} + \left(\frac{\Delta}{2ab}\right)^{2/3} \leq 1.$$

With $\Delta, \Delta' > 0$, we then have $(\Delta/2\Delta')^{2/3} < 1$ and $(\Delta/2ab)^{2/3} < 1$. So, $\Delta/2\Delta' < 1$, and therefore, $\mathit{\Delta}' > \mathit{\Delta}/2$, as desired. ■

The process of proving the corresponding result for the hyperbola is very similar to that of the ellipse and so is left to the reader.

CHAPTER 3

COMPLEX AND MATRIX INEQUALITIES

Our final chapter will focus on complex and matrix inequalities. This will include inequalities arising from univalent functions and the study of the very famous Bieberbach Conjecture, as well as one of Jacques Hadamard's matrix inequalities involving determinants. We include these inequalities here because they are interesting and give excellent examples of the applications of inequalities in advanced Mathematics. For the study of univalent functions, there are several references, but we include Duren [Dur83] and Nehari [Neh52]. Also, the notes of Professor K. T. Hahn are helpful. For matrix inequalities two excellent references are Marcus and Minc [MM64] and Graham [Gra87].

3.1 A brief history of the Bieberbach Conjecture

In an article in *The American Mathematical Monthly*, J. Korevaar outlines the history of the Bieberbach Conjecture [Kor86]. It began around 1900 when Hilbert and Osgood reproved Riemann's Conformal Mapping Theorem. A mapping $\omega = f(z)$ is conformal in a domain D if it is analytic (differentiable in D) and the derivative of $f(z)$ is never zero in D. It is well known that univalent (one to one) functions that are analytic in a domain D must be conformal in D. Riemann's

Theorem states that if U is a simply connected domain in the complex plane $\mathbb{C}$ (not equal to $\mathbb{C}$), then there is a conformal mapping from the unit disk D: $\{|z| < 1\}$ onto U.

Around 1910, conformal mapping became a popular topic for German mathematicians. Some of their study focused on functions f that are not only analytic and univalent (or "schlicht") in the unit disk D, but also have the properties that $f(0) = 0$ and $f'(0) = 1$. We let S denote this specific class of functions. Then, every f in S has a power series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

Paul Koebe is famous for studying a specific function in S. This function, which is properly named the Koebe function, is defined to be $k(z) = z/(1-z)^2$. Notice

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

and so,

$$\left(\frac{1}{1-z}\right)^2 = 1 + 2z + 3z^2 + 4z^3 + \cdots.$$

Therefore,

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots.$$

It is not difficult to show that $k(z)$ is in S and $a_n = n$ for all n.

In 1916, another German mathematician, Ludwig Bieberbach, proved that for every f in S, $|a_2| \leq 2$. As a footnote, he proposed that perhaps for every f in S and every n, $|a_n| \leq n$. This became known as the famous Bieberbach Conjecture, and was not proven for sixty-eight years. However, before the proof of the conjecture, there were many milestones reached. In 1923, C. Löwner proved that $|a_3| \leq 3$ for every f in S. Between 1955 and 1972, many arduous proofs were developed for the cases when n = 4, 5, and 6. In 1925, J. Littlewood found that as n approaches infinity, $|a_n| < en$. FitzGerald improved upon this in 1972, and his student Horowitz somewhat bettered it in 1978. They found that $|a_n| < 1.07n$. Finally, in 1984, Louis de Branges of Purdue University successfully proved the Bieberbach Conjecture, in addition to even stronger results about the class S. Now the conjecture is known as the Bieberbach – de Branges Theorem.

3.2. Univalent functions

We will now explore some monumental results in the study of univalent functions and the quest toward a proof of the Bieberbach Conjecture. First will be the Area Theorem, whose discovery we owe to the Swedish mathematician T. Grönwall, in 1914. For this theorem, we are concerned with functions g that are univalent and analytic in the unit disk D except for a simple pole at the origin. We call this class of functions Σ. Then, every $g \in \Sigma$ has the following Laurent series expansion:

$$g(z) = \frac{1}{z} + b_o + b_1 z + b_2 z^2 + \cdots$$

We will establish the following lemma before proving the Area Theorem.

Lemma 3.2.1. *Suppose C is a positively oriented any simple closed contour, P is the region bounded by the contour, and A_P is the area of that region. Suppose also that $z = x + iy$ is any point in the z-plane. Then,*

$$A_P = \frac{1}{2i}\oint_C \bar{z}dz.$$

Proof. Green's Theorem states that if C is a positively oriented, piecewise smooth, simple closed contour in the plane, P is the region bounded by C, and L and M are functions of x and y with continuous partial derivatives, then

$$\oint_C (Ldx + Mdy) = \iint_P \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy. \tag{3.2.1}$$

So, for any complex number $z = x + iy$,

$$\begin{aligned}
\frac{1}{2i}\oint_C \bar{z}dz &= \frac{1}{2i}\oint_C (x - iy)(dx + idy) \\
&= \frac{1}{2i}\oint_C [(xdx + ydy) + i(xdy - ydx)] \\
&= \frac{1}{2i}\oint_C (xdx + ydy) + \frac{1}{2}\oint_C (xdy - ydx) \\
&= \frac{1}{2i}\iint_P (0 - 0)dxdy + \frac{1}{2}\iint_P (1 + 1)dxdy \text{, by (3.2.1)} \\
&= \frac{1}{2}\iint_P 2dxdy \\
&= \iint_P dxdy \\
&= A_P,
\end{aligned}$$

where we used a known result from Calculus. This completes our proof. ∎

We are now ready to prove our first main result, the Area Theorem, which appears in Duren [Dur83] as well as the notes of Professor K.T. Hahn.

Theorem 3.2.2. (The Area Theorem) *Let $g \in \Sigma$. Then,*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Proof. Let $g \in \Sigma$. Then g has the form

$$g(z) = \frac{1}{z} + b_o + b_1 z + b_2 z^2 + \cdots$$

$$= \frac{1}{z} + \sum\nolimits_{m=0}^{\infty} b_m z^m.$$

where $z \in D$. Then, $\omega = g(z)$ will map D onto the exterior of some domain, with $g(0) = \infty$ (because g has a simple pole at $z = 0$). We now consider D_r, where $D_r = \{z: |z| < r < 1\}$. Call A_r the area of the image domain of D_r under $\omega = g(z)$ and γ_r the image of the boundary of D_r. Then, by Lemma 3.2.1 applied in the ω-plane, we have

$$A_r = -\frac{1}{2i}\int_{\gamma_r} \bar{\omega} d\omega$$

$$= -\frac{1}{2i}\int_0^{2\pi} \overline{g(z)} \cdot g'(z) dz,$$

since $\omega = g(z)$. Also, because of the $1/z$ term in $g(z)$, γ_r is in the clockwise direction and hence negatively oriented. Therefore, to make the area A_r positive, we multiply by –1 (see Sansone and Gerretsen [SG69]). Now, with

$$g(z) = \frac{1}{z} + \sum\nolimits_{m=0}^{\infty} b_m z^m,$$

we have

$$g'(z) = -\frac{1}{z^2} + \sum_{m=1}^{\infty} m b_m z^{m-1}$$

and

$$\overline{g(z)} = \frac{1}{\bar{z}} + \sum_{n=0}^{\infty} \overline{b_n z^n}\,.$$

Letting $z = re^{i\theta}$ gives us

$$g\left(re^{i\theta}\right) = \frac{1}{re^{i\theta}} + \sum_{m=0}^{\infty} b_m r^m e^{im\theta},$$

$$g'(re^{i\theta}) = -\frac{1}{r^2 e^{2i\theta}} + \sum_{m=1}^{\infty} m b_m r^{m-1} e^{i(m-1)\theta},$$

and

$$\overline{g(re^{\iota\theta})} = \frac{e^{i\theta}}{r} + \sum_{n=0}^{\infty} \overline{b_n} r^n e^{-in\theta}.$$

Therefore,

$$A_r = -\frac{1}{2i}\int_0^{2\pi} \overline{g(z)} \cdot g'(z)dz$$

$$= -\frac{1}{2i}\int_0^{2\pi} \overline{g(re^{\iota\theta})} \cdot g'\left(re^{i\theta}\right) ire^{i\theta} d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \left(\frac{e^{i\theta}}{r} + \sum_{n=0}^{\infty} \overline{b_n} r^n e^{-in\theta}\right)\left(-\frac{1}{r^2 e^{2i\theta}} + \sum_{m=1}^{\infty} m b_m r^{m-1} e^{i(m-1)\theta}\right)\left(re^{i\theta}\right) d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \left(\frac{e^{i\theta}}{r} + \sum_{n=0}^{\infty} \overline{b_n} r^n e^{-in\theta}\right)\left(-\frac{1}{re^{i\theta}} + \sum_{m=1}^{\infty} m b_m r^m e^{im\theta}\right) d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\left(-\frac{1}{r^2} + \sum_{m=1}^{\infty} mb_m r^{m-1} e^{i(m+1)\theta} - \sum_{n=0}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta}\right.$$

$$\left. + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} mb_m \overline{b_n} r^{m+n} e^{i(m-n)\theta}\right) d\theta. \tag{3.2.2}$$

Since

$$\int_0^{2\pi} e^{i(j-k)\theta}\, d\theta = \begin{cases} 0, \text{if } j \neq k \\ 2\pi, \text{if } j = k \end{cases},$$

then

$$\int_0^{2\pi} \sum_{m=1}^{\infty} mb_m r^{m-1} e^{i(m+1)\theta} d\theta = 0$$

and

$$\int_0^{2\pi} \sum_{n=0}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta} d\theta = 0.$$

Also

$$\int_0^{2\pi} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} mb_m \overline{b_n} r^{m+n} e^{i(m-n)\theta} d\theta = 0,$$

except when $m = n$. These, together with (3.2.2.) gives us

$$A_r = -\frac{1}{2}\left[-\frac{1}{r^2}(2\pi) + \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\,(2\pi)\right]$$

$$= \pi\left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\right)$$

$$\geq 0.$$

Therefore,

$$\frac{1}{r^2} \geq \sum_{n=1}^{\infty} n|b_n|^2 r^{2n},$$

where $0 < r < 1$. Then, if we take the limit as r approaches 1, we have our desired result:

$$1 \geq \sum_{n=1}^{\infty} n|b_n|^2. \quad \blacksquare$$

As mentioned above, in 1916 L. Bieberbach proved that $|a_2| \leq 2$. We shall prove his result below, after first establishing the following lemma [Dur83].

Lemma 3.2.3. *Suppose $f(z) \in S$. Then,*

$$|a_2^2 - a_3| \leq 1.$$

Equality holds for the Koebe function.

Proof. Suppose $f(z) \in S$. Then, f can be written as

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots.$$

So,

$$\frac{1}{f(z)} = \frac{1}{z + a_2 z^2 + a_3 z^3 + \cdots}$$

$$= \frac{1}{z} - a_2 + (a_2^2 - a_3)z - \cdots,$$

by simple long division. Therefore, $1/f(z)$ is an element of Σ and we can apply the Area Theorem (Theorem 3.2.2). By this theorem we have, for $n = 1$,

$$|b_1| \leq 1,$$

where b_1 denotes the coefficient of z. Therefore,

$$|a_2^2 - a_3| \leq 1.$$

Finally, in the Koebe function, notice that

$$|a_2^2 - a_3| = |2^2 - 3| = 1. \quad \blacksquare$$

Theorem 3.2.4. (Bieberbach's Theorem) *Suppose $f(z) \in S$. Then,*

$$|a_2| \leq 2,$$

where equality holds for the Koebe function.

Proof. Suppose $f(z) \in S$. Again, f can be written

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

Consider $h(z) = \sqrt{f(z^2)}$. Note that since $h(z)$ is a composition of analytic functions, it is also analytic in the unit disk. To see that $h(z)$ is univalent, we will first show that it is an odd function. Notice that

$$\begin{aligned} h(z) &= \sqrt{z^2 + a_2 z^4 + a_3 z^6 + \cdots} \\ &= \sqrt{z^2(1 + a_2 z^2 + a_3 z^4 + \cdots)} \\ &= z\sqrt{1 + a_2 z^2 + a_3 z^4 + \cdots}. \end{aligned} \tag{3.2.3}$$

Now, the binomial theorem for n not an integer states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

This, together with (3.2.3) yields

$$h(z) = z\left[1 + \frac{1}{2}(a_2 z^2 + a_3 z^4 + \cdots) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}(a_2 z^2 + a_3 z^4 + \cdots)^2 \right.$$

$$\left. + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}(a_2 z^2 + a_3 z^4 + \cdots)^3 + \cdots\right]$$

$$= z\left[1 + \frac{1}{2}(a_2z^2 + a_3z^4 + \cdots) - \frac{1}{8}(a_2z^2 + a_3z^4 + \cdots)^2\right.$$

$$\left. + \frac{1}{16}(a_2z^2 + a_3z^4 + \cdots)^3 - \cdots\right]$$

$$= z\left[1 + \left(\frac{a_2}{2}\right)z^2 + \left(\frac{a_3}{2} - \frac{a_2^2}{8}\right)z^4 + \cdots\right]$$

$$= \left[z + \left(\frac{a_2}{2}\right)z^3 + \left(\frac{a_3}{2} - \frac{a_2^2}{8}\right)z^5 + \cdots\right]. \tag{3.2.4}$$

From the result above we see that $h(z)$ is odd. With this, we can show that $h(z)$ is univalent. Suppose $h(z_1) = h(z_2)$. Then

$$\sqrt{f(z_1^2)} = \sqrt{f(z_2^2)}.$$

So,

$$f(z_1^2) = f(z_2^2).$$

Since f is in S, f is univalent. Therefore,

$$z_1^2 = z_2^2,$$

which implies

$$z_1 = -z_2$$

or

$$z_1 = z_2.$$

Suppose $z_1 = -z_2$. Then

$$h(z_1) = h(-z_2).$$

From (3.2.4), h is an odd function. So,

$$h(-z_2) = -h(z_2).$$

But, we assumed $h(z_1) = h(z_2)$. Therefore,

$$-h(z_2) = h(z_2),$$

which implies that $h(z) = 0$ for every z in D. This is not the case. Hence, $z_1 \neq -z_2$, leaving us with $z_1 = z_2$. So, $h(z)$ is one-to-one.

Notice also that $h(0) = 0$ and $h'(0) = 1$. This, along with $h(z)$ being analytic and univalent in the unit disk, tells us that h is in S. Therefore, we can apply Lemma 3.2.3 and get

$$\left|(0)^2 - \frac{a_2}{2}\right| \leq 1.$$

Hence,

$$|a_2| \leq 2,$$

as desired. ■

For an application of Bieberbach's Theorem that $|a_2| \leq 2$, we look at Koebe's One-Quarter Theorem, which he discovered in as early as 1907 [Dur83].

Theorem 3.2.5. (Koebe One-Quarter Theorem) *For every function f in S and unit disk D, the images $f(D)$ contain the disk $D_{1/4} = \{\omega : |\omega| < 1/4\}$, and this result is sharp.*

Proof. Suppose $f \in S$ with ω_0 not in the image of $\omega = f(z)$. Let

$$\begin{aligned} g(z) &= \frac{f(z)}{1 - [f(z)/\omega_0]} \\ &= f(z)\left\{\frac{1}{1 - [f(z)/\omega_0]}\right\} \\ &= f(z)\left\{1 + \frac{f(z)}{\omega_0} + \left[\frac{f(z)}{\omega_0}\right]^2 + \left[\frac{f(z)}{\omega_0}\right]^3 + \cdots\right\}, \end{aligned}$$

by the infinite geometric sum formula. Since $f \in S$, it has the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots.$$

So,

$$g(z) = [z + a_2 z^2 + \cdots]\left[1 + \left(\frac{z + a_2 z^2 + \cdots}{\omega_0}\right) + \left(\frac{z + a_2 z^2 + \cdots}{\omega_0}\right)^2 + \cdots\right]$$

$$= z + \left(\frac{1}{\omega_0} + a_2\right) z^2 + \cdots,$$

and we can see that g is analytic in the unit disk since it is differentiable in the entire domain *D.* It is easy to show that g is also univalent, $g(0) = 0$, and $g'(0) = 1$. So, $g \in S$, and we can apply Bieberbach's Theorem (Theorem 3.2.4). This, together with the Triangle Inequality gives

$$\left|\frac{1}{\omega_0}\right| - |a_2| \leq \left|\frac{1}{\omega_0} + a_2\right| \leq 2.$$

Therefore,

$$\left|\frac{1}{\omega_0}\right| \leq 2 + |a_2|,$$

and so,

$$|\omega_0| \geq \frac{1}{4}.$$

Since we defined ω_0 to be any point not in the image $\omega = f(z)$, then for all f in S, the image of f contains an open disk, $|\omega| < 1/4$. ■

The following theorem proves the Bieberbach Conjecture for real coefficients [Neh52].

Theorem 3.2.6. *If the coefficients a_n of a function $f \in S$ are real, then*

$$|a_n| \leq n,$$

where again, equality holds with the Koebe function.

Proof. Suppose $f \in S$. Then, f is univalent in the unit disk. So,

$$f(z_1) - f(z_2) \neq 0 \tag{3.2.5}$$

for distinct points z_1 and z_2. Consider $z_1 = re^{i\theta}$ and $z_2 = re^{-i\theta}$. Then,

$$\begin{aligned} f(z_1) - f(z_2) &= (z_1 + a_2 z_1^2 + a_3 z_1^3 + \cdots) - (z_2 + a_2 z_2^2 + a_3 z_2^3 + \cdots) \\ &= (z_1 - z_2) + a_2(z_1^2 - z_2^2) + a_3(z_1^3 - z_2^3) + \cdots \\ &= r\left(e^{i\theta} - e^{-i\theta}\right) + a_2 r^2\left(e^{2i\theta} - e^{-2i\theta}\right) + a_3 r^3\left(e^{3i\theta} - e^{-3i\theta}\right) + \cdots \\ &= \sum_{n=1}^{\infty} a_n r^n\left(e^{in\theta} - e^{-in\theta}\right) \\ &= 2i \sum_{n=1}^{\infty} a_n r^n \sin(n\theta) \\ &\neq 0, \end{aligned}$$

by (3.2.5). Therefore,

$$\sum_{n=1}^{\infty} a_n r^n \sin(n\theta) \neq 0.$$

Since $\sin\theta \neq 0$ for $0 < \theta < \pi$, we can also say

$$p(\theta) = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta) \sin\theta \neq 0$$

for $0 < \theta < \pi$. Because the coefficients a_n are real, so is $p(\theta)$. Also, $p(\theta)$ is continuous. These facts, together with $p(\theta) \neq 0$, imply that

$$p(\theta) < 0 \quad \text{or} \quad p(\theta) > 0$$

for $0 < \theta < \pi$. In addition, since $-\sin\theta = \sin(-\theta)$, we have $p(\theta) = p(-\theta)$. So, $p(\theta)$ is an even function, which means

$$p(\theta) \le 0 \quad \text{or} \quad p(\theta) \ge 0 \tag{3.2.6}$$

for $0 \le \theta < 2\pi$.

Now, using the following well-known identity,

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)],$$

we obtain

$$\begin{aligned}
p(\theta) &= \frac{1}{2}\sum_{n=1}^{\infty} a_n r^n \{\cos[(n-1)\theta] - \cos[(n+1)\theta]\} \\
&= \frac{1}{2}\left\{\sum_{n=1}^{\infty} a_n r^n \cos[(n-1)\theta] - \sum_{n=1}^{\infty} a_n r^n \cos[(n+1)\theta]\right\} \\
&= \frac{1}{2}\left\{a_1 r + a_2 r^2 \cos\theta + \sum_{n=3}^{\infty} a_n r^n \cos[(n-1)\theta] - \sum_{n=1}^{\infty} a_n r^n \cos[(n+1)\theta]\right\} \\
&= \frac{r}{2}\left\{a_1 + a_2 r\cos\theta + \sum_{n=3}^{\infty} a_n r^{n-1} \cos[(n-1)\theta] - \sum_{n=1}^{\infty} a_n r^{n-1} \cos[(n+1)\theta]\right\}.
\end{aligned}$$

Let $m = n - 1$ in the first sum above and $m = n + 1$ in the second. Then,

$$\begin{aligned}
p(\theta) &= \frac{r}{2}\left[a_1 + a_2 r\cos\theta + \sum_{m=2}^{\infty} a_{m+1} r^m \cos(m\theta) - \sum_{m=2}^{\infty} a_{m-1} r^{m-2} \cos(m\theta)\right] \\
&= \frac{r}{2}\left[1 + a_2 r\cos\theta + \sum_{m=2}^{\infty}\left(a_{m+1} - \frac{a_{m-1}}{r^2}\right) r^m \cos(m\theta)\right].
\end{aligned} \tag{3.2.7}$$

(With $f \in S$, we know that $a_1 = 1$.) So, since $\int \cos\theta d\theta = \sin\theta$ and $\sin\theta = \sin(2\pi) = 0$,

$$\int_0^{2\pi} p(\theta)d\theta = \frac{r}{2}(2\pi) = \pi r. \tag{3.2.8}$$

Therefore, $\int_0^{2\pi} p(\theta)d\theta \geq 0$ for $0 \leq \theta < 2\pi$. Since the integral is nonnegative and we know, by (3.2.6), that $p(\theta) \geq 0$ or $p(\theta) \leq 0$, this implies $p(\theta) \geq 0$. And so, since $1 \pm \cos(n\theta)$ is also nonnegative,

$$\begin{aligned}
0 &\leq \int_0^{2\pi} p(\theta)[1 \pm \cos(n\theta)]d\theta \\
&= \int_0^{2\pi} p(\theta)d\theta \pm \int_0^{2\pi} p(\theta) \cdot cos(n\theta)\, d\theta \\
&= \pi r \pm \int_0^{2\pi} p(\theta) \cdot cos(n\theta)\, d\theta\,, \quad \text{by (3.2.8)} \\
&= \pi r \pm \frac{r}{2}\int_0^{2\pi} \Bigg[1 + a_2 r\cos\theta \\
&\quad + \sum_{m=2}^{\infty}\left(a_{m+1} - \frac{a_{m-1}}{r^2}\right) r^m \cos(m\theta)\Bigg]\cos(n\theta)d\theta, \quad \text{by (3.2.7)} \\
&= \pi r \pm \frac{r}{2}\Bigg[\int_0^{2\pi} \cos(n\theta)d\theta + \int_0^{2\pi} a_2 r\cos\theta\cos(n\theta)d\theta + \\
&\quad + \sum_{m=2}^{\infty}\left(a_{m+1} - \frac{a_{m-1}}{r^2}\right) r^m \int_0^{2\pi} \cos(m\theta)\cos(n\theta)d\theta\Bigg].
\end{aligned} \tag{3.2.9}$$

The orthogonality of the cosine function on the interval $[0,2\pi)$ gives us

$$\int_0^{2\pi} \cos(m\theta)\cos(n\theta)\, d\theta = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases}.$$

This, together with (3.2.9), yields

$$0 \leq \pi r \pm \frac{r}{2}\left(a_{n+1} - \frac{a_{n-1}}{r^2}\right) r^n \pi$$

for $n \geq 2$ (since the sum in (3.2.9) began at 2). So,

$$\pi r \pm \frac{\pi}{2}\left(a_{n+1} - \frac{a_{n-1}}{r^2}\right) r^{n+1} \geq 0,$$

and

$$2 \pm \left(a_{n+1} - \frac{a_{n-1}}{r^2}\right) r^n \geq 0.$$

Therefore,

$$\left|\left(a_{n+1} - \frac{a_{n-1}}{r^2}\right) r^n\right| \leq 2,$$

or,

$$\left|a_{n+1} - \frac{a_{n-1}}{r^2}\right| r^n \leq 2,$$

for $n \geq 2$. Because the above is true for $0 < r < 1$, we take the limit as r approaches 1. Then,

$$|a_{n+1} - a_{n-1}| \leq 2.$$

So, by the Triangle Inequality [Bul98],

$$\big||a_{n+1}| - |a_{n-1}|\big| \leq |a_{n+1} - a_{n-1}| \leq 2,$$

and therefore

$$|a_{n+1}| - |a_{n-1}| \leq 2. \tag{3.2.10}$$

Finally, we will use induction along with (3.2.10) to obtain $|a_n| \leq n$ for all real values of n. Suppose first that n is odd.

<u>Base Step</u>: Consider $n = 1$. It is known that $|a_1| = 1$.

<u>Induction Step</u>: Let $n = 2k + 1$ for positive integer k. Suppose that our inequality holds for these odd values of n. That is, $|a_{2k+1}| \leq 2k + 1$. Well, by (3.2.10),

$$|a_{2k+3}| - |a_{2k+1}| \leq 2.$$

Therefore,

$$|a_{2k+3}| \leq |a_{2k+1}| + 2 \leq (2k+1) + 2 = 2k+3,$$

by our induction hypothesis. This proves $|a_n| \leq n$ for n odd. Now, suppose that n is even.

Base Step: Consider $n = 2$. Well, $|a_2| \leq 2$ by Bieberbach's Theorem, Theorem 3.2.4.

Induction Step: Let $n = 2k$ for positive integer k. Suppose that our inequality holds for these even values of n. That is, $|a_{2k}| \leq 2k$. But, by (3.2.10),

$$|a_{2k+2}| - |a_{2k}| \leq 2.$$

Therefore,

$$|a_{2k+2}| \leq |a_{2k}| + 2 \leq (2k) + 2,$$

by our induction hypothesis. This proves $|a_n| \leq n$ for n even. Hence, for all real values of n, $|a_n| \leq n$. ■

3.3 Hadamard's Inequality for real matrices

Now we will move our discussion to matrices. The focus of this section and the following is a famous matrix inequality that was discovered by French mathematician Jacques Hadamard. His inequality, from 1893 [Had93], states that for any $n \times n$ square matrix $A = [a_{ij}]$,

$$|\det A|^2 \leq \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right).$$

In this section, we will prove Hadamard's Inequality for the special case of A being a real matrix. We first establish the following definitions and lemmas [Axl97].

Definition 3.3.1. A set $(\boldsymbol{v_1}, \ldots, \boldsymbol{v_n})$ of vectors in a vector space V is *linearly independent* if $a_1\boldsymbol{v_1} + \cdots + a_n\boldsymbol{v_n} = 0$ for $a_1, \ldots, a_n \in \mathbb{C}$ implies $a_1 = \cdots = a_n = 0$. Otherwise, the set is *linearly dependent.*

Definition 3.3.2. The set of all linear combinations of $(\boldsymbol{v_1}, \ldots, \boldsymbol{v_n})$ is called the *span* of $(\boldsymbol{v_1}, \ldots, \boldsymbol{v_n})$. That is, $\text{span}(\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}) = \{a_1\boldsymbol{v_1} + \cdots + a_n\boldsymbol{v_n}\colon a_1, \ldots, a_n \in \mathbb{C}\}$.

Definition 3.3.3. A *basis* of a vector space V is a linearly independent spanning set.

Definition 3.3.4. The *norm* of a vector $\boldsymbol{v}$ is defined to be the measure of its size or length. We will denote the norm of vector $\boldsymbol{v}$ by $\|\boldsymbol{v}\|$.

Definition 3.3.5. The *Euclidean norm* of a vector $\boldsymbol{v}$ in $\mathbb{C}^n$ is defined to be $\|\boldsymbol{v}\|^2 = \sum_{i=1}^{n}|\boldsymbol{v_i}|^2$, where $\boldsymbol{v} = (v_1\ v_2\ \ldots\ v_n)$.

Definition 3.3.6. The *inner product* of two vectors $\boldsymbol{u} = (u_1 \ldots u_n)$ and $\boldsymbol{v} = (v_1 \ldots v_n)$ is denoted $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$. For our purposes, we will define the inner product as $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}\boldsymbol{v}^* = u_1\overline{v_1} + \cdots + u_n\overline{v_n}$ (where $\boldsymbol{v}^*$ is the conjugate transpose of $\boldsymbol{v}$.) Also, note that $\|\boldsymbol{u}\|^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle$.

Definition 3.3.7. Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are *orthogonal* if and only if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$.

Definition 3.3.8. A set of vectors is called *orthonormal* if the vectors are pairwise orthogonal and each vector has a norm equal to one. In other words, a set

$(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ of vectors is orthonormal if $\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = 0$ for $i \neq j$ and $\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = 1$ for $i = j$ (where $i, j = 1, \dots, n$).

Definition 3.3.9. An *orthonormal basis* of a vector space V is an orthonormal list of vectors that is also a basis (linearly independent spanning set) for V.

Lemma 3.3.10. *Suppose $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ is an orthonormal basis of a vector space V. Then for every $\boldsymbol{v} \in V$,*

$$\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{e_1} \rangle \boldsymbol{e_1} + \cdots + \langle \boldsymbol{v}, \boldsymbol{e_n} \rangle \boldsymbol{e_n}.$$

Proof. Because $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ is a basis for V, for every $\boldsymbol{v} \in V$, there exist scalars $a_1, \dots, a_n \in \mathbb{C}$ such that

$$\boldsymbol{v} = a_1 \boldsymbol{e_1} + \cdots + a_n \boldsymbol{e_n}. \tag{3.3.1}$$

Taking the inner product of both sides with $\boldsymbol{e_j}$ (where $1 \leq j \leq n$), we have

$$\begin{aligned} \langle \boldsymbol{v}, \boldsymbol{e_j} \rangle &= \langle a_1 \boldsymbol{e_1} + \cdots + a_n \boldsymbol{e_n}, \boldsymbol{e_j} \rangle \\ &= \langle a_1 \boldsymbol{e_1}, \boldsymbol{e_j} \rangle + \cdots + \langle a_j \boldsymbol{e_j}, \boldsymbol{e_j} \rangle + \cdots + \langle a_n \boldsymbol{e_n}, \boldsymbol{e_j} \rangle \\ &= a_1 \langle \boldsymbol{e_1}, \boldsymbol{e_j} \rangle + \cdots + a_j \langle \boldsymbol{e_j}, \boldsymbol{e_j} \rangle + \cdots + a_n \langle \boldsymbol{e_n}, \boldsymbol{e_j} \rangle, \end{aligned}$$

by additivity and homogeneity of the inner product. Because $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ is orthonormal, we have $\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = 0$ for $i \neq j$ and $\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = 1$ for $i = j$ (where $i, j = 1, \dots, n$). So,

$$\langle \boldsymbol{v}, \boldsymbol{e_j} \rangle = a_1(0) + \cdots + a_j(1) + \cdots + a_n(0) = a_j.$$

This, together with (3.3.1) gives us

$$\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{e_1} \rangle \boldsymbol{e_1} + \cdots + \langle \boldsymbol{v}, \boldsymbol{e_n} \rangle \boldsymbol{e_n}. \quad \blacksquare$$

Lemma 3.3.11. *Suppose* $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ *is an orthonormal basis of a vector space V. Then for every* $\boldsymbol{v} \in V$*,*

$$\|\boldsymbol{v}\|^2 = |\langle \boldsymbol{v}, \boldsymbol{e_1} \rangle|^2 + \cdots + |\langle \boldsymbol{v}, \boldsymbol{e_n} \rangle|^2.$$

Proof. Let $\boldsymbol{v} \in V$. Then, by Lemma 3.3.10,

$$\boldsymbol{v} = \alpha_1 \boldsymbol{e_1} + \cdots + \alpha_n \boldsymbol{e_n},$$

where $\alpha_k = \langle \boldsymbol{v}, \boldsymbol{e_k} \rangle$ for $k = 1, \dots, n$. So,

$$\begin{aligned}
\|\boldsymbol{v}\|^2 &= \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\
&= \langle \alpha_1 \boldsymbol{e_1} + \cdots + \alpha_n \boldsymbol{e_n}, \alpha_1 \boldsymbol{e_1} + \cdots + \alpha_n \boldsymbol{e_n} \rangle \\
&= \langle \alpha_1 \boldsymbol{e_1}, \alpha_1 \boldsymbol{e_1} \rangle + \langle \alpha_1 \boldsymbol{e_1}, \alpha_2 \boldsymbol{e_2} \rangle + \cdots + \langle \alpha_1 \boldsymbol{e_1}, \alpha_n \boldsymbol{e_n} \rangle \\
&+ \\
&\vdots \\
&+ \langle \alpha_n \boldsymbol{e_n}, \alpha_1 \boldsymbol{e_1} \rangle + \langle \alpha_n \boldsymbol{e_n}, \alpha_2 \boldsymbol{e_2} \rangle + \cdots + \langle \alpha_n \boldsymbol{e_n}, \alpha_n \boldsymbol{e_n} \rangle \\
&= \alpha_1 \overline{\alpha_1} \langle \boldsymbol{e_1}, \boldsymbol{e_1} \rangle + \cdots + \alpha_1 \overline{\alpha_1} \langle \boldsymbol{e_1}, \boldsymbol{e_n} \rangle \\
&+ \\
&\vdots \\
&+ \alpha_n \overline{\alpha_n} \langle \boldsymbol{e_n}, \boldsymbol{e_1} \rangle + \cdots + \alpha_n \overline{\alpha_n} \langle \boldsymbol{e_n}, \boldsymbol{e_n} \rangle,
\end{aligned}$$

again by additivity and homogeneity of the inner product. And, because $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ are orthonormal, we get

$$\begin{aligned}
\|\boldsymbol{v}\|^2 &= \alpha_1 \overline{\alpha_1} + \cdots + \alpha_n \overline{\alpha_n} \\
&= |\alpha_1|^2 + \cdots + |\alpha_n|^2 \\
&= |\langle \boldsymbol{v}, \boldsymbol{e_1} \rangle|^2 + \cdots + |\langle \boldsymbol{v}, \boldsymbol{e_n} \rangle|^2. \quad \blacksquare
\end{aligned}$$

Lemma 3.3.12. (Gram-Schmidt) *If $(\boldsymbol{v_1}, \dots, \boldsymbol{v_n})$ is a linearly independent set of vectors in V, then there exists an orthonormal set of vectors $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ in V such that*

$$\operatorname{span}\{\boldsymbol{v_1}, \dots, \boldsymbol{v_m}\} = \operatorname{span}\{\boldsymbol{e_1}, \dots, \boldsymbol{e_m}\},$$

for each $m = 1,2, \dots, n$.

Proof: See Axler [Axl97].

We are now ready to prove our first Theorem – Hadamard's Inequality for real matrices. We will provide two proofs. The first uses properties of Linear Algebra [Had] and the second is a geometric approach [Bel43].

Theorem 3.3.13. (Hadamard's Inequality for real matrices) *Let $A = [a_{ij}]$* be any *$n \times n$ matrix with real entries. Then,*

$$(\det A)^2 \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}^2\right).$$

Proof 1. Suppose $(\boldsymbol{a_1}, \dots, \boldsymbol{a_n})$ is a set of column vectors in $\mathbb{R}^n$ with $A = (\boldsymbol{a_1}\ \boldsymbol{a_2} \dots\ \boldsymbol{a_n})$ the corresponding $n \times n$ real matrix. If the set were linearly dependent, then $\det A$ would equal zero, and the above inequality is trivial. Suppose then, the columns of A are linearly independent. Then, by Gram-Schmidt (Lemma 3.3.12), an orthonormal basis $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ for $\mathbb{R}^n$ exists and $\operatorname{span}\{\boldsymbol{a_1}, \dots, \boldsymbol{a_m}\} = \operatorname{span}\{\boldsymbol{e_1}, \dots, \boldsymbol{e_m}\}$, for each $m = 1,2, \dots, n$. By Lemma 3.3.10, every $\boldsymbol{a_j}$ for $1 \leq j \leq n$ can be written

$$\boldsymbol{a_j} = \sum_{i=1}^{n} \langle \boldsymbol{a_j}, \boldsymbol{e_i} \rangle \boldsymbol{e_i}.$$

But, span$\{\boldsymbol{a_1}, \dots, \boldsymbol{a_m}\}$ = span$\{\boldsymbol{e_1}, \dots, \boldsymbol{e_m}\}$ tells us that each $\boldsymbol{a_j}$ has a shorter expansion of the form

$$\boldsymbol{a_j} = \sum_{i=1}^{j} \langle \boldsymbol{a_j}, \boldsymbol{e_i} \rangle \boldsymbol{e_i}. \tag{3.3.2}$$

Now, let $B = [b_{kl}]$ be the $n \times n$ upper triangular matrix such that

$$b_{kl} = \langle \boldsymbol{a_l}, \boldsymbol{e_k} \rangle \text{ if } 1 \le k \le l$$

and

$$b_{kl} = 0 \text{ if } l < k \le n.$$

That is,

$$B = \begin{bmatrix} \langle \boldsymbol{a_1}, \boldsymbol{e_1} \rangle & \langle \boldsymbol{a_2}, \boldsymbol{e_1} \rangle & \cdots & \langle \boldsymbol{a_n}, \boldsymbol{e_1} \rangle \\ 0 & \langle \boldsymbol{a_2}, \boldsymbol{e_2} \rangle & \cdots & \langle \boldsymbol{a_n}, \boldsymbol{e_2} \rangle \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \langle \boldsymbol{a_n}, \boldsymbol{e_n} \rangle \end{bmatrix}.$$

Let $E = [\boldsymbol{e_1}\ \boldsymbol{e_2}\ \cdots\ \boldsymbol{e_n}]$ be the $n \times n$ matrix whose columns correspond to the orthonormal basis $(\boldsymbol{e_1}, \dots, \boldsymbol{e_n})$ for $\mathbb{R}^n$. Notice,

$$EB = [\boldsymbol{e_1}\ \boldsymbol{e_2}\ \cdots\ \boldsymbol{e_n}] \begin{bmatrix} \langle \boldsymbol{a_1}, \boldsymbol{e_1} \rangle & \langle \boldsymbol{a_2}, \boldsymbol{e_1} \rangle & \cdots & \langle \boldsymbol{a_n}, \boldsymbol{e_1} \rangle \\ 0 & \langle \boldsymbol{a_2}, \boldsymbol{e_2} \rangle & \cdots & \langle \boldsymbol{a_n}, \boldsymbol{e_2} \rangle \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \langle \boldsymbol{a_n}, \boldsymbol{e_n} \rangle \end{bmatrix}$$

$$= [\boldsymbol{e_1} \langle \boldsymbol{a_1}, \boldsymbol{e_1} \rangle \quad \boldsymbol{e_1} \langle \boldsymbol{a_2}, \boldsymbol{e_1} \rangle + \boldsymbol{e_2} \langle \boldsymbol{a_2}, \boldsymbol{e_2} \rangle \quad \dots \quad \boldsymbol{e_1} \langle \boldsymbol{a_n}, \boldsymbol{e_1} \rangle + \cdots + \boldsymbol{e_n} \langle \boldsymbol{a_n}, \boldsymbol{e_n} \rangle]$$

$$= [\langle \boldsymbol{a_1}, \boldsymbol{e_1} \rangle \boldsymbol{e_1} \quad \langle \boldsymbol{a_2}, \boldsymbol{e_1} \rangle \boldsymbol{e_1} + \langle \boldsymbol{a_2}, \boldsymbol{e_2} \rangle \boldsymbol{e_2} \quad \dots \quad \langle \boldsymbol{a_n}, \boldsymbol{e_1} \rangle \boldsymbol{e_1} + \cdots + \langle \boldsymbol{a_n}, \boldsymbol{e_n} \rangle \boldsymbol{e_n}],$$

since each inner product is simply a scalar. So,

$$EB = \left[\sum_{i=1}^{1} \langle \boldsymbol{a_1}, \boldsymbol{e_i} \rangle \boldsymbol{e_i} \quad \sum_{i=1}^{2} \langle \boldsymbol{a_2}, \boldsymbol{e_i} \rangle \boldsymbol{e_i} \quad \cdots \quad \sum_{i=1}^{n} \langle \boldsymbol{a_n}, \boldsymbol{e_i} \rangle \boldsymbol{e_i}. \right]$$

$$= [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \quad \boldsymbol{a_n}],$$

by (3.3.2.). Hence,

$$A = EB.$$

So,

$$\begin{aligned}
(\det A)^2 &= \det A \cdot \det A \\
&= \det A^T \cdot \det A, \\
&= \det(A^T A) \\
&= \det[(EB)^T EB]\text{, since } A = EB \\
&= \det(B^T E^T EB) \\
&= \det(B^T E^{-1} EB)\text{, since } E \text{ is orthogonal} \\
&= \det(B^T B) \\
&= \det B^T \cdot \det B \\
&= \det B \cdot \det B \\
&= (\det B)^2 \\
&= \left(\prod_{j=1}^{n} \langle \boldsymbol{a_j}, \boldsymbol{e_j} \rangle\right)^2\text{, since } B \text{ is upper triangular} \\
&= \prod_{j=1}^{n} \langle \boldsymbol{a_j}, \boldsymbol{e_j} \rangle^2 \\
&\le \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \langle \boldsymbol{a_j}, \boldsymbol{e_i} \rangle^2\right),
\end{aligned}$$

since one element of a sum of nonnegative numbers will be less than or equal to the entire sum. Therefore,

$$(\det A)^2 \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{n}\langle \boldsymbol{a_j}, \boldsymbol{e_i}\rangle^2\right), \tag{3.3.3}$$

Now, by Lemma 3.3.11,

$$\prod_{j=1}^{n}\left(\sum_{i=1}^{n}\langle \boldsymbol{a_j}, \boldsymbol{e_i}\rangle^2\right) = \prod_{j=1}^{n}\left\|\boldsymbol{a_j}\right\|^2 = \prod_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}^2\right)$$

This, together with (3.3.3) gives us

$$(\det A)^2 \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}^2\right),$$

which completes our proof. ■

The technique used in our next proof can be extended to higher dimensions to obtain Hadamard's Inequality for real matrices in the general case. For simplicity, we will restrict our discussion here to the special cases when A is 2×2 and 3×3 [Bel43].

Proof 2. Before we begin, note that $\det A = \det A^T$. So, proving Hadamard's Inequality for the sums of the squares of the row entries is equivalent to doing the same for the columns. Therefore, we can equivalently prove

$$(\det A)^2 \leq \prod_{i=1}^{n}\left(\sum_{j=1}^{n}\left(a_{ij}\right)^2\right). \tag{3.3.4}$$

Suppose first that A is a 2×2 matrix with real entries. Let

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.$$

Notice then that proving (3.3.4) for a 2×2 matrix is equivalent to proving

$$(a_1b_2 - a_2b_1)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2). \tag{3.3.5}$$

We begin by considering any two points in the plane, $A = (a_1, a_2)$ and $B = (b_1, b_2)$, with the origin, O. Now, let $C = (-a_2, a_1)$. Then, line OA has equation $y = (a_2/a_1)x$ and line OC has equation $y = -(a_1/a_2)x$. Hence, lines OA and OC are perpendicular.

Let $m\angle AOB = \theta$. Consider ΔABO as in Figure 3.1 below.

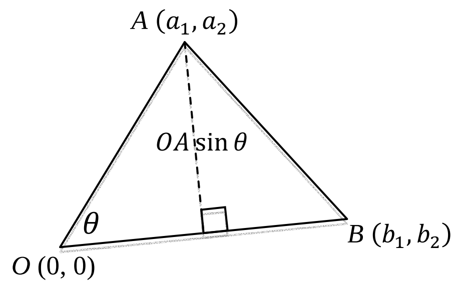

Figure 3.1: ΔABO with O at the origin

Denote the area of ΔABO as Δ_1. Then,

$$\Delta_1 = \frac{1}{2} OB \cdot OA \cdot \sin\theta = \frac{1}{2}\sqrt{b_1^2 + b_2^2} \cdot \sqrt{a_1^2 + a_2^2} \cdot \sin\theta$$

$$= \frac{\sin\theta}{2}\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}\,. \tag{3.3.6}$$

It is known we can also express the area of ΔABO using determinants. That is,

$$\Delta_1 = \left|\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right|. \tag{3.3.7}$$

Next, Consider ΔCBO with height h, as in Figure 3.2 on the following page.

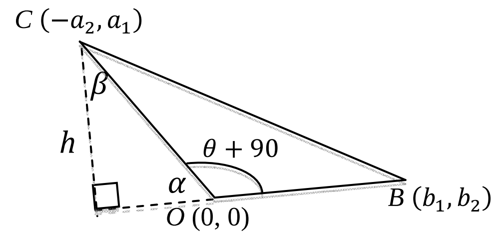

Figure 3.2: ΔCBO with O at the origin

Since OA and OC are perpendicular and $m\angle AOB = \theta$, we have $m\angle BOC = \theta + 90$. Therefore, ΔCBO is obtuse and so the altitude from C lies outside the triangle, as in Figure 3.2. Now,

$$m\angle\alpha = 180 - (\theta + 90) = 90 - \theta,$$

and so

$$m\angle\beta = 90 - \alpha = 90 - (90 - \theta) = \theta.$$

Therefore,

$$h = OC \cdot \cos\beta = OC \cdot \cos\theta.$$

Then, if we let Δ_2 denote the area of ΔCBO,we have

$$\Delta_2 = \frac{1}{2} OB \cdot OC \cdot \cos\theta = \frac{1}{2}\sqrt{b_1^2 + b_2^2} \cdot \sqrt{a_2^2 + a_1^2} \cdot \cos\theta$$

$$= \frac{\cos\theta}{2}\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} \qquad (3.3.8)$$

Again, we can also express the area of ΔCBO using determinants. So,

$$\Delta_2 = \left|\frac{1}{2}\begin{vmatrix} b_1 & b_2 \\ -a_2 & a_1 \end{vmatrix}\right|. \tag{3.3.9}$$

Now, by (3.3.6) and (3.3.8) we have

$$(\Delta_1)^2 + (\Delta_2)^2 = \frac{\sin^2\theta}{4}(a_1^2 + a_2^2)(b_1^2 + b_2^2) + \frac{\cos^2\theta}{4}(a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

$$= \frac{(\sin^2\theta + \cos^2\theta)}{4}(a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

$$= \frac{1}{4}(a_1^2 + a_2^2)(b_1^2 + b_2^2). \tag{3.3.10}$$

And, from (3.3.7) and (3.3.9) we have

$$(\Delta_1)^2 + (\Delta_2)^2 = \left(\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)^2 + \left(\frac{1}{2}\begin{vmatrix} b_1 & b_2 \\ -a_2 & a_1 \end{vmatrix}\right)^2$$

$$= \frac{1}{4}[(a_1b_2 - a_2b_1)^2 + (a_1b_1 + a_2b_2)^2]$$

$$\geq \frac{1}{4}(a_1b_2 - a_2b_1)^2. \tag{3.3.11}$$

Hence, by (3.3.10) and (3.3.11),

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1b_2 - a_2b_1)^2,$$

which is precisely (3.3.5) and therefore Hadamard's Inequality for real 2×2 matrices.

Suppose now that A is any real 3×3 matrix, where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Then, the following is equivalent to proving (3.3.4) for $n = 3$:

$$(\det A)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2). \tag{3.3.12}$$

Consider the tetrahedron $OABC_1$ with vertices $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C_1 = (c_1, c_2, c_3)$, and origin O. Let the lengths of $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC_1}$ be a, b, and c, respectively. Also, call h_1 the altitude from C_1 to the plane OAB and θ the angle formed by OA and OB. If V_1 is the volume of tetrahedron $OABC_1$, then

$$\begin{aligned} V_1 &= \frac{1}{3}\left(\frac{1}{2}ba\sin\theta\right)h_1 \\ &= \frac{1}{6}abh_1\sin\theta. \end{aligned} \tag{3.3.13}$$

It is known that we can also express the volume of the tetrahedron in the following determinant form:

$$V_1 = \left|\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right|. \tag{3.3.14}$$

Now, consider two other points, C_2 and C_3, such that $OC_1 = OC_2 = OC_3 = c$ and $\overrightarrow{OC_1}$, $\overrightarrow{OC_2}$, and $\overrightarrow{OC_3}$ are mutually perpendicular. Similar to above, we call h_2 the altitude from C_2 to the plane OAB and h_3 the altitude from C_3 to the plane OAB. Then, if V_2 is the volume of tetrahedron $OABC_2$ and V_3 is the volume of tetrahedron $OABC_3$, we have

$$V_2 = \frac{1}{6}abh_2\sin\theta \tag{3.3.15}$$

and

$$V_3 = \frac{1}{6}abh_3\sin\theta. \tag{3.3.16}$$

So, by (3.3.13) – (3.3.16) we have that

$$\frac{1}{36}(\det A)^2 = \frac{1}{36}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$

$$= V_1^2$$

$$\leq V_1^2 + V_2^2 + V_3^2$$

$$= \frac{1}{36}(h_1^2 + h_2^2 + h_3^2)a^2b^2\sin^2\theta. \qquad (3.3.17)$$

Since $\overrightarrow{OC_1}$, $\overrightarrow{OC_2}$, and $\overrightarrow{OC_3}$ are mutually perpendicular we can, without loss of generality, let C_1 be on the x-axis, C_2 be on the y-axis, and C_3 be on the z-axis. Then $C_1 = (c, 0,0)$, $C_2 = (0, c, 0)$, and $C_3 = (0,0, c)$. Consider the parallelepiped formed by OA, OB, and OC_1, with height h_1 and parallelogram base. Since the volume of the parallelepiped is equal to the area of its base times the height, we have

$$h_1 = \left|\frac{\left|\left(\overrightarrow{OA}\times\overrightarrow{OB}\right)\cdot\overrightarrow{OC_1}\right|}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right| = \left|\frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c & 0 & 0 \end{vmatrix}}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right| = \left|\frac{c\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right|.$$

Similarly,

$$h_2 = \left|\frac{\left|\left(\overrightarrow{OA}\times\overrightarrow{OB}\right)\cdot\overrightarrow{OC_2}\right|}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right| = \left|\frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & c & 0 \end{vmatrix}}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right| = \left|\frac{-c\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right|$$

and

$$h_3 = \left|\frac{\left|\left(\overrightarrow{OA}\times\overrightarrow{OB}\right)\cdot\overrightarrow{OC_3}\right|}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right| = \left|\frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & c \end{vmatrix}}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right| = \left|\frac{c\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}{\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|}\right|.$$

So,

$$h_1^2 + h_2^2 + h_3^2 = \frac{c^2}{\left\|\overrightarrow{OA} \times \overrightarrow{OB}\right\|^2}\left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2\right) = c^2.$$

Notice that we used that the sum of the squares of the determinants is equal to the square of $\left\|\overrightarrow{OA} \times \overrightarrow{OB}\right\|$. Then, (3.3.17) becomes

$$\begin{aligned} \frac{1}{36}(\det A)^2 &\leq \frac{1}{36} a^2 b^2 c^2 \sin^2\theta \\ &\leq \frac{1}{36} a^2 b^2 c^2, \end{aligned}$$

since $\sin^2\theta \leq 1$ for all values of θ. And, with a as the length of OA, b the length of OB, and c the length of OC_1,

$$a^2 b^2 c^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2).$$

Therefore,

$$(\det A)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2),$$

which is (3.3.12) and Hadamard's Inequality for a 3×3 real matrix. ■

3.4 Hadamard's Inequality for complex matrices

Hadamard's Inequality in fact holds for any complex square matrix. We will present a surprisingly simple proof. To do so, we need the following definitions and lemmas.

Definition 3.4.1. A square matrix A is *Hermitian* if $A^* = A$, where A^* represents the conjugate transpose of A.

Definition 3.4.2. A Hermitian matrix is *positive semidefinite* if, for any nonzero vector $\boldsymbol{x}$, it is the case that $\boldsymbol{x}^* A \boldsymbol{x} \geq 0$. A Hermitian matrix A is *positive definite* if the inequality is strict for any nonzero vector $\boldsymbol{x}$.

Definition 3.4.3. An *eigenvector* of a square matrix A is a nonzero vector $\boldsymbol{x}$ that satisfies the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where λ, called the *eigenvalue*, is a scalar.

Lemma 3.4.4. *All the eigenvalues of a Hermitian matrix are real.*

Proof: See Graham [Gra87].

Lemma 3.4.5. *If A is an $n \times n$ Hermitian matrix, then A is positive semidefinite if and only if the eigenvalues of A are nonnegative.*

Proof: See Graham [Gra87].

Lemma 3.4.6. *Let A be a square matrix with complex entries. Then, the eigenvalues of $A^* A$ are nonnegative.*

Proof. Using a property of the inner product,

$$\langle A^* A\boldsymbol{v}, \boldsymbol{v}\rangle = \langle A\boldsymbol{v}, A\boldsymbol{v}\rangle = \|A\boldsymbol{v}\|^2 \geq 0,$$

which implies

$$\langle A^* A\boldsymbol{v}, \boldsymbol{v}\rangle \geq 0, \tag{3.4.1}$$

for any vector $\boldsymbol{v} \in \mathbb{C}^{\boldsymbol{n}}$. Suppose then that $\boldsymbol{v}$ is an eigenvector of $A^* A$ with corresponding eigenvalue λ. That is, $A^* A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then,

$$\langle A^* A\boldsymbol{v}, \boldsymbol{v}\rangle = \langle \lambda\boldsymbol{v}, \boldsymbol{v}\rangle = \lambda\langle \boldsymbol{v}, \boldsymbol{v}\rangle.$$

Thus,

$$\lambda = \frac{\langle A^* A\boldsymbol{v}, \boldsymbol{v}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}\rangle}.$$

Because $\boldsymbol{v}$ is an eigenvector, it is nonzero. Hence $\langle \boldsymbol{v}, \boldsymbol{v}\rangle > 0$. This, together with (3.4.1) tells us that $\lambda \geq 0$. Therefore, the eigenvalues of $A^* A$ are nonnegative. ■

Lemma 3.4.7. *For any square matrix, the determinant is equal to the product of the eigenvalues.*

Proof: See Graham [Gra87].

Lemma 3.4.8. *For any square matrix A with complex entries,*

$$|\det(A)|^2 = \det(A^* A).$$

Proof. First we see that

$$\begin{aligned} \det(A^* A) &= (\det A^*)(\det A) \\ &= (\det \bar{A})(\det A), \text{since} \det A^T = \det A \\ &= \overline{(\det A)}(\det A) \\ &= |\det(A)|^2. \quad ■ \end{aligned}$$

Lemma 3.4.9. *Suppose A is a positive semidefinite $n \times n$ matrix with complex entries and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Suppose also that $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n} \in \mathbb{C}^n$ is a set of orthonormal vectors. Then,*

$$\prod_{j=1}^{k} \lambda_{n-j+1} \leq \prod_{j=1}^{k} \langle A\boldsymbol{e_j}, \boldsymbol{e_j}\rangle,$$

for $1 \leq k \leq n$.

Proof: See Marcus and Minc [MM64].

We are now ready to prove Hadamard's Inequality for complex matrices [MM64].

Theorem 3.4.10. (Hadamard's Inequality for complex matrices) For any $n \times n$ square matrix $A = [a_{ij}]$ with each $a_{ij} \in \mathbb{C}$,

$$|\det A|^2 \leq \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right).$$

Proof. Notice that A^*A is Hermitian since $(A^*A)^* = A^*A$. By Lemma 3.4.6, the eigenvalues of A^*A are nonnegative, and so A^*A is positive semidefinite by Lemma 3.4.5. Therefore, we can apply Lemma 3.4.9 to A^*A. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be the eigenvalues of A^*A, and suppose that $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n} \in \mathbb{C}^{\mathrm{n}}$ is the standard set of orthonormal vectors. Then,

$$\begin{aligned} |\det A|^2 &= \det(A^*A)\text{, by Lemma 3.4.8} \\ &= \prod_{j=1}^{n} \lambda_{n-j+1}\text{, by Lemma 3.4.7} \\ &\leq \prod_{j=1}^{n} \langle A^*A\boldsymbol{e_j}, \boldsymbol{e_j} \rangle\text{, by Lemma 3.4.9} \\ &= \prod_{j=1}^{n} \langle A\boldsymbol{e_j}, A\boldsymbol{e_j} \rangle. \end{aligned}$$

And so,

$$|\det A|^2 \le \prod_{j=1}^{n} \langle A\boldsymbol{e_j}, A\boldsymbol{e_j} \rangle. \tag{3.4.2}$$

Notice that

$$A\boldsymbol{e_j} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ a_{j1} & \cdots & a_{jj} & \cdots & a_{jn} \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the 1 entry in $\boldsymbol{e_j}$ appears in the jth position. So,

$$A\boldsymbol{e_j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = A^{(j)},$$

with $A^{(j)}$ denoting the jth column of A. Hence,

$$\prod_{j=1}^{n} \langle A\boldsymbol{e_j}, A\boldsymbol{e_j} \rangle = \prod_{j=1}^{n} \langle A^{(j)}, A^{(j)} \rangle$$

$$= \prod_{j=1}^{n} \left\| A^{(j)} \right\|^2$$

$$= \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \left| a_{ij} \right|^2 \right).$$

This, together with (3.4.2) yields

$$|\det A|^2 \le \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \left| a_{ij} \right|^2 \right). \quad \blacksquare$$

CONCLUSION

In this thesis, we explored several mathematical inequalities, but have only scratched the surface. The vast subject continues to be an area of study, and researchers persistently work for more answers and applications. For instance, although Katsuura introduced a chain inequality for the sine and tangent of an acute angle, many more extensions are unknown. Also, we proved the following result of Katsuura and Obaid: If z is a complex number with modulus not equal to one and n is a positive integer, then

$$\left|\frac{z^n - 1}{z - 1}\right| \leq \frac{|z|^n - 1}{|z| - 1}.$$

We need further investigation to determine if the following is true for any non-zero values of m:

$$\left|\frac{z^n - z^m}{z - 1}\right| \leq \frac{|z|^n - |z|^m}{|z| - 1}.$$

Many inequalities that generalize Price's Inequality are also still to be found. Similarly, although a significant amount of inequalities about univalent functions have been discovered, it is certainly possible to do more in this area (see Duren [Dur83]), and the same is true for matrix inequalities. As with other branches of Mathematics, new discoveries beg additional questions, which hopefully sparks more research and investigation.

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